B Online Appendix for "Making Decisions under Model Misspecification"

The Online Appendix is structured as follows. In Section B.1, we prove the ancillary results we use in deriving our main representation results. Section B.2 regroups the proofs of the main representation results in the body of the paper. Section B.3 contains all the remaining proofs. Specifically, in Section B.3.1, we prove all the results about misspecification attitudes and neutrality (so those pertaining to Sections 4.2 and 4.3). Section B.3.2 includes the proofs of the other results that appear in the body of the paper (Propositions 1, 6, and 8 as well as Corollary 3). Section B.3.3 is devoted to the proofs of the results in Appendix A. In the final Section B.4, we provide some additional material discussed informally in the main text. We first show the irrelevance of convexity in the entropic model for the set Q (Section B.4.1). We conclude by providing the axiomatization of our criterion with only one set Q (Section B.4.2).

In all appendices, we denote by $B_0(\Sigma)$ the space of Σ-measurable simple functions $\varphi : S \to \mathbb{R}$, endowed with the supnorm $\| \|_\infty$. Given an interval T in $\mathbb{R}$, we denote by $B_0(\Sigma, T)$ the subset of $B_0(\Sigma)$ consisting of all functions φ that take values in T. The norm dual of $B_0(\Sigma)$ can be identified with the space $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). Given a subset $C \subseteq \Delta$, the *effective domain* of $f : C \to (-\infty, \infty]$, denoted by $\operatorname{dom} f$, is the set $\{p \in C : f(p) < \infty\}$ where f takes finite values. Recall that the function f is *grounded* if the infimum of its image is 0, i.e., $\inf_C f = 0$. With the usual abuse of notation, throughout the paper, we denote by k both the real number and the constant function taking value k.

B.1 Ancillary results for the main representation results

We here prove the two ancillary variational lemmas we will use in proving Theorem 1.

Lemma 4 *Let $\succsim$ be a variational preference represented by $V : \mathcal{F} \to \mathbb{R}$ defined by*

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + c(p) \right\} \qquad \forall f \in \mathcal{F}$$

and let $\bar{p} \in \Delta$. If $\succsim$ is unbounded, then the following conditions are equivalent:

(i) $c(\bar{p}) = 0$*;*

(ii) $x_f^{\bar{p}} \succsim f$ *for all* $f \in \mathcal{F}$*;*

(iii) for each $f \in \mathcal{F}$ *and for each* $x \in X$

$$x \succ x_f^{\bar{p}} \implies x \succ f$$

Proof We actually prove that (i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii), with equivalence when $\succsim$ is unbounded.

(i) implies (ii). Let $f \in \mathcal{F}$. It is enough to observe that $c(\bar{p}) = 0$ implies

$$V\left(x_f^{\bar{p}}\right) = u\left(x_f^{\bar{p}}\right) = \int u(f)\,\mathrm{d}\bar{p} + c(\bar{p}) \geq \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p)\right\} = V(f)$$

yielding that $x_f^{\bar{p}} \succsim f$.

(ii) implies (iii). Assume that $x_f^{\bar{p}} \succsim f$ for all $f \in \mathcal{F}$. Since $\succsim$ is complete and transitive, it follows that if $x \succ x_f^{\bar{p}}$, then $x \succ f$.

(iii) implies (ii). By contradiction, suppose that there exists $f \in \mathcal{F}$ such that $f \succ x_f^{\bar{p}}$. Let $x_f \in X$ be such that $x_f \sim f$. This implies that $x_f \succ x_f^{\bar{p}}$ and so $x_f \succ f$, a contradiction.

(ii) implies (i). Let $\succsim$ be unbounded. Assume that $x_f^{\bar{p}} \succsim f$ for all $f \in \mathcal{F}$, i.e., $V(f) \leq \int u(f)\,\mathrm{d}\bar{p}$ for all $f \in \mathcal{F}$. So, $\bar{p}$ corresponds to a SEU preference that is less ambiguity averse than $\succsim$. By Lemma 32 of Maccheroni et al. (2006), we can conclude that $c(\bar{p}) = 0$. ■

We denote by $\Delta^{\ll}(Q)$ the collection of all probabilities p which are absolutely continuous with respect to Q, that is, if $A \in \Sigma$ and $q(A) = 0$ for all $q \in Q$, then $p(A) = 0$.

Lemma 5 *Let $\succsim$ be a variational preference represented by $V : \mathcal{F} \to \mathbb{R}$ defined by*

$$V(f) = \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p)\right\} \qquad \forall f \in \mathcal{F}$$

If $\succsim$ is unbounded, then the following conditions are equivalent:

(i) For each $f, g \in \mathcal{F}$

$$f \overset{Q}{=} g \implies f \sim g$$

(ii) $\operatorname{dom} c \subseteq \Delta^{\ll}(Q)$.

Proof We begin by observing that in proving the two implications, Q being either compact or convex plays no role.

(i) implies (ii). Let $p \in \Delta \backslash \Delta^{\ll}(Q)$. It follows that there exists $A \in \Sigma$ such that $q(A) = 0$ for all $q \in Q$ as well as $p(A) > 0$. Define $I : B_0(\Sigma) \to \mathbb{R}$ by $I(\varphi) = \min_{p\in\Delta}\left\{\int \varphi dp + c(p)\right\}$ for all $\varphi \in B_0(\Sigma)$. Since u is unbounded, for each $\lambda \in \mathbb{R}$ there exists $x_\lambda \in X$ such that $u(x_\lambda) = \lambda$. Similarly, there exists $y \in X$ such that $u(y) = 0$. For each $\lambda \in \mathbb{R}$ define $f_\lambda = x_\lambda A y$. By construction, we have that $f_\lambda \overset{Q}{=} y$ for all $\lambda \in \mathbb{R}$. This implies that $I(\lambda 1_A) = V(f_\lambda) = V(y) = I(0) = 0$ for all $\lambda \in \mathbb{R}$. By Maccheroni et al. (2006) and since u is unbounded and $p(A) > 0$, we have that

$$c(p) = \sup_{\varphi\in B_0(\Sigma)}\left\{I(\varphi) - \int \varphi dp\right\} \geq \sup_{\lambda\in\mathbb{R}}\{I(\lambda 1_A) - \lambda p(A)\} = \infty$$

Since p was arbitrarily chosen, it follows that $\operatorname{dom} c \subseteq \Delta^{\ll}(Q)$.

(ii) implies (i). Assume that $\operatorname{dom} c \subseteq \Delta^{\ll}(Q)$. If $f \overset{Q}{=} g$, then $u(f) \overset{Q}{=} u(g)$. This implies that $u(f) \overset{p}{=} u(g)$ for all $p \in \Delta^{\ll}(Q)$ and, in particular,

$$\begin{aligned} V(f) &= \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p)\right\} = \min_{p\in\Delta^{\ll}(Q)}\left\{\int u(f)\,\mathrm{d}p + c(p)\right\} \\ &= \min_{p\in\Delta^{\ll}(Q)}\left\{\int u(g)\,\mathrm{d}p + c(p)\right\} = \min_{p\in\Delta}\left\{\int u(g)\,\mathrm{d}p + c(p)\right\} = V(g) \end{aligned}$$

proving that $f \sim g$. ■

B.2 Proofs of the main representation results

In this appendix, we provide the proofs of our representation results (Theorem 1 and Proposition 7).

Proof of Theorem 1 We only prove (i) implies (ii), the converse being routine.[36] We proceed by steps.

Step 1. $\succsim_Q^*$ agrees with $\succsim_{Q'}^*$ on X for all $Q, Q' \in \mathcal{Q}$. In particular, there exists an affine and onto function $u : X \to \mathbb{R}$ representing $\succsim_Q^*$ on X for all $Q \in \mathcal{Q}$.

Proof of the Step Let $Q, Q' \in \mathcal{Q}$ be such that $Q \supseteq Q'$. Note that $\succsim_Q^*$ and $\succsim_{Q'}^*$, restricted to X, satisfy weak order, continuity and risk independence.[37] By Herstein and Milnor (1953) and since $\succsim_Q^*$ and $\succsim_{Q'}^*$ are non-trivial, there exist two non-constant affine functions $u_Q, u_{Q'} : X \to \mathbb{R}$ which represent $\succsim_Q^*$ and $\succsim_{Q'}^*$, respectively. Since $\{\succsim_Q^*\}_{Q\in\mathcal{Q}}$ is monotone in model ambiguity, we have that

$$u_Q(x) \geq u_Q(y) \Longrightarrow u_{Q'}(x) \geq u_{Q'}(y)$$

By Corollary B.3 of Ghirardato et al. (2004), u_Q and $u_{Q'}$ are equal up to an affine and positive transformation, proving that $\succsim_Q^*$ and $\succsim_{Q'}^*$ agree on X. Next, fix $\bar{q} \in \Delta^\sigma$. Set $u = u_{\bar{q}}$. Given any other $q \in \Delta^\sigma$, consider $\bar{Q} \in \mathcal{Q}$ such that $\bar{Q} \supseteq \{\bar{q}, q\}$. By the previous part, it follows that $u_{\bar{Q}}$, u_q and $u_{\bar{q}}$ are equal up to an affine and positive transformation. Given that q was arbitrarily chosen, we can set $u = u_q$ for all $q \in Q$. Similarly, given a generic $Q \in \mathcal{Q}$, select $q \in Q$. Since $Q \supseteq \{q\}$, it follows that we can set $u = u_Q$, proving the main part of the statement. By Lemma

[36] The only exception is the proof that the representation implies subjective Q-coherence. This is a consequence of Theorem 2.4.18 in Zalinescu (2002) paired with Lemma 32 of Maccheroni et al. (2006).

[37] To prove that $\succsim_Q^*$ satisfies risk independence, it suffices to deploy the same technique of Lemma 28 of Maccheroni et al. (2006) and observe that $\succsim_Q^*$ is complete and transitive, that is a weak order, on X. This yields that

$$x \sim_Q^* y \implies \frac{1}{2}x + \frac{1}{2}z \sim_Q^* \frac{1}{2}y + \frac{1}{2}z \quad \forall z \in X$$

By Theorem 2 of Herstein and Milnor (1953) and since $\succsim_Q^*$ satisfies continuity, we can conclude that $\succsim_Q^*$ satisfies risk independence.

59 of Cerreia-Vioglio et al. (2011b) and since $\succsim_Q^*$ is non-trivial and unbounded for all $Q \in \mathcal{Q}$, we can conclude that u is onto. $\square$

Step 2. For each $q \in \Delta^\sigma$ there exists a normalized, monotone, translation invariant and concave functional $\hat{I}_q : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim_q^* g \Longleftrightarrow \hat{I}_q(u(f)) \geq \hat{I}_q(u(g)) \tag{40}$$

Moreover, there exists a unique grounded, lower semicontinuous and convex function $c_q : \Delta \to [0,\infty]$ such that

$$\hat{I}_q(\varphi) = \min_{p \in \Delta} \left\{ \int \varphi dp + c_q(p) \right\} \qquad \forall \varphi \in B_0(\Sigma) \tag{41}$$

Proof of the Step Fix $q \in \Delta^\sigma$. Since $\succsim_q^*$ is an unbounded dominance relation which is complete, we have that $\succsim_q^*$ is axiomatically a variational preference. By the proof of Theorem 3 and Proposition 6 of Maccheroni et al. (2006) and Step 1, there exists an onto and affine function $u_q : X \to \mathbb{R}$, which can be set to be equal to u, and, given u, a unique, grounded, lower semicontinuous and convex function $c_q : \Delta \to [0,\infty]$ such that (40) and (41) hold. $\square$

Define $c : \Delta \times \Delta^\sigma \to [0,\infty]$ by $c(p,q) = c_q(p)$ for all $(p,q) \in \Delta \times \Delta^\sigma$.

Step 3. For each $Q \in \mathcal{Q}$ we have that $f \succsim_Q^* g$ if and only if $f \succsim_q^* g$ for all $q \in Q$. In particular, we have that

$$f \succsim_Q^* g \iff \min_{p \in \Delta} \left\{ \int u(f)\, \mathrm{d}p + c(p,q) \right\} \geq \min_{p \in \Delta} \left\{ \int u(g)\, \mathrm{d}p + c(p,q) \right\} \quad \forall q \in Q \tag{42}$$

Proof of the Step Fix $Q \in \mathcal{Q}$. Since $\{\succsim_Q^*\}_{Q \in \mathcal{Q}}$ is monotone in model ambiguity, we have that

$$f \succsim_Q^* g \implies f \succsim_q^* g \qquad \forall q \in Q$$

Since $\{\succsim_Q^*\}_{Q \in \mathcal{Q}}$ is separable, we can conclude that $f \succsim_Q^* g$ if and only if $f \succsim_q^* g$ for all $q \in Q$. By Step 2 and the definition of c, (42) follows. $\square$

Step 4. $\succsim_Q^*$ agrees with $\succsim_Q$ on X for all $Q \in \mathcal{Q}$. Moreover, $\succsim_Q$ is represented on X by the function u of Step 1.

Proof of the Step Fix $Q \in \mathcal{Q}$. Note that $\succsim_Q^*$ and $\succsim_Q$, restricted to X, satisfy weak order, continuity and risk independence. By Herstein and Milnor (1953) and since $\succsim_Q$ is non-trivial, there exists a non-constant affine function v_Q which represents $\succsim_Q$. By Step 1, $\succsim_Q^*$ is represented by u. Since $(\succsim_Q^*, \succsim_Q)$ jointly satisfy consistency, it follows that

$$u(x) \geq u(y) \Longrightarrow v_Q(x) \geq v_Q(y)$$

By Corollary B.3 of Ghirardato et al. (2004), v_Q and u are equal up to an affine and positive transformation. So we can set $v_Q = u$, proving the statement. $\square$

Step 5. For each $Q \in \mathcal{Q}$ there exists a normalized and monotone functional $I_Q : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim_Q g \Longleftrightarrow I_Q(u(f)) \geq I_Q(u(g))$$

Moreover, for each $q \in \Delta^\sigma$ we have that $I_q = \hat{I}_q$ and, in particular, $\succsim_q^*$ coincides with $\succsim_q$.

Proof of the Step Fix $Q \in \mathcal{Q}$.[38] By Step 4, $\succsim_Q$ is represented on X by the onto and affine function u of Step 1. Since $\succsim_Q$ is solvable, for each $f \in \mathcal{F}$ there exists $x_{f,Q} \in X$ such that $f \sim_Q x_{f,Q}$. Since $\operatorname{Im} u = \mathbb{R}$, we have that $B_0(\Sigma) = \{u(f) : f \in \mathcal{F}\}$. Define $I_Q : B_0(\Sigma) \to \mathbb{R}$ by $I_Q(\varphi) = u(x_{f,Q})$ where $f \in \mathcal{F}$ is such that $u(f) = \varphi$. Since $\succsim_Q$ is a complete, transitive and monotone binary relation, we have that I_Q is well defined and monotone. Moreover, by construction, we have that $I_Q(k1_S) = k$ for all $k \in \mathbb{R}$. By construction, note that

$$I_Q(u(f)) \geq I_Q(u(g)) \iff u(x_{f,Q}) \geq u(x_{g,Q}) \iff x_{f,Q} \succsim_Q x_{g,Q} \iff f \succsim_Q g$$

Next, fix $q \in \Delta^\sigma$. By Step 2 and the previous part of the proof, we have that $f \mapsto \hat{I}_q(u(f))$ and $f \mapsto I_q(u(f))$ represent, respectively, $\succsim_q^*$ and $\succsim_q$. Since $(\succsim_q^*, \succsim_q)$ jointly satisfy consistency and the range of both functionals is $\mathbb{R}$, we can conclude that there exists a (not necessarily strictly) monotone function $h : \mathbb{R} \to \mathbb{R}$ such that $I_q(u(f)) = h\left(\hat{I}_q(u(f))\right)$ for all $f \in \mathcal{F}$. Since I_q and $\hat{I}_q$ are normalized and $\operatorname{Im} u = \mathbb{R}$, we have that $h(u(x)) = u(x)$ for all $x \in X$, proving that h is the identity. Since $q \in \Delta^\sigma$ was arbitrarily chosen, it follows that $I_q = \hat{I}_q$ and, in particular, $\succsim_q^*$ coincides with $\succsim_q$ for all $q \in \Delta^\sigma$. $\square$

Step 6. $c(p,q) = 0$ if and only if $p = q$.

Proof of the Step By Steps 2 and 5, we have that $I_q = \hat{I}_q$ and $\succsim_q^*$ coincides with $\succsim_q$ for all $q \in \Delta^\sigma$. By Lemma 4 and since $\succsim_q$ is subjectively $\{q\}$-coherent, we have that $\operatorname{argmin} c(\cdot, q) = \operatorname{argmin} c_q = \{q\}$. $\square$

Step 7. $\operatorname{dom} c(\cdot, q) \subseteq \Delta^{\ll}(q)$ *for all* $q \in \Delta^\sigma$.[39]

Proof of the Step By Step 2 and Lemma 5 and since $\succsim_q^*$ is objectively $\{q\}$-coherent, we can conclude that $\operatorname{dom} c(\cdot, q) \subseteq \Delta^{\ll}(q)$ for all $q \in \Delta^\sigma$. $\square$

Step 8. c is jointly lower semicontinuous.

Proof of the Step Define the map $J : B_0(\Sigma) \times \Delta^\sigma \to \mathbb{R}$ by $J(\varphi, q) = \hat{I}_q(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $q \in \Delta^\sigma$. Observe that, for each $(p,q) \in \Delta \times \Delta^\sigma$,

$$c(p,q) = c_q(p) = \sup_{\varphi \in B_0(\Sigma)} \left\{ \hat{I}_q(\varphi) - \int \varphi dp \right\} = \sup_{\varphi \in B_0(\Sigma)} \left\{ J(\varphi, q) - \int \varphi dp \right\} \tag{43}$$

We begin by observing that J is lower semicontinuous in the second argument. Note that for

[38] We follow the strategy proof of Proposition 1 in Cerreia-Vioglio et al. (2011a).
[39] The set $\Delta^{\ll}(q)$ contains all p in Δ such that if $A \in \Sigma$ and $q(A) = 0$, then $p(A) = 0$.

each $\varphi \in B_0(\Sigma)$ and for each $q \in \Delta^\sigma$

$$J(\varphi, q) = \hat{I}_q(\varphi) = u(x_{f,q}) \qquad \text{where } f \in \mathcal{F} \text{ is s.t. } \varphi = u(f)$$

Fix $\varphi \in B_0(\Sigma)$ and $t \in \mathbb{R}$. By the axiom of lower semicontinuity, the set

$$\{q \in \Delta^\sigma : J(\varphi, q) \leq t\} = \{q \in \Delta^\sigma : u(x) \geq u(x_{f,q})\} = \left\{q \in \Delta^\sigma : x \succsim_q^* x_{f,q}\right\}$$

is closed where $x \in X$ and $f \in \mathcal{F}$ are such that $u(x) = t$ as well as $u(f) = \varphi$. Since φ and t were arbitrarily chosen, this yields that J is lower semicontinuous in the second argument. Since J is lower semicontinuous in the second argument, the map $(p, q) \mapsto J(\varphi, q) - \int \varphi dp$, defined over $\Delta \times \Delta^\sigma$, is jointly lower semicontinuous for all $\varphi \in B_0(\Sigma)$. By (43) and the definition of c, we conclude that c is jointly lower semicontinuous. $\square$

Step 9. $I_Q(\varphi) \geq \inf_{q \in Q} \hat{I}_q(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $Q \in \mathcal{Q}$.

Proof of the Step Fix $Q \in \mathcal{Q}$ and $\varphi \in B_0(\Sigma)$. Since each $\hat{I}_q$ is normalized and monotone and u is onto, we have that $\hat{I}_q(\varphi) \in [\min_{s \in S} \varphi(s), \max_{s \in S} \varphi(s)] \subseteq \operatorname{Im} u = \mathbb{R}$ for all $q \in Q$. Since $\varphi \in B_0(\Sigma)$, it follows that there exists $f \in \mathcal{F}$ such that $\varphi = u(f)$ and $x \in X$ such that $u(x) = \inf_{q \in Q} \hat{I}_q(\varphi)$. Note that $\hat{I}_{q'}(u(f)) = \hat{I}_{q'}(\varphi) \geq \inf_{q \in Q} \hat{I}_q(\varphi) = u(x) = \hat{I}_{q'}(u(x))$ for all $q' \in Q$. By Steps 2 and 3, $f \succsim_Q^* x$. Since $\left(\succsim_Q^*, \succsim_Q\right)$ jointly satisfy consistency, we have that $f \succsim_Q x$. By Step 5, this implies that $I_Q(\varphi) = I_Q(u(f)) \geq I_Q(u(x)) = u(x) = \inf_{q \in Q} \hat{I}_q(\varphi)$, proving the step. $\square$

Step 10. $I_Q(\varphi) \leq \inf_{q \in Q} \hat{I}_q(\varphi)$ for all $\varphi \in B_0(\Sigma)$ and for all $Q \in \mathcal{Q}$.

Proof of the Step Fix $Q \in \mathcal{Q}$ and $\varphi \in B_0(\Sigma)$. We use the same objects and notation of Step 9. For each $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $u(x_\varepsilon) = u(x) + \varepsilon$. By Steps 2 and 3 and since $\inf_{q \in Q} \hat{I}_q(\varphi) = u(x)$, it follows that for each $\varepsilon > 0$ there exists $q \in Q$ such that $\hat{I}_q(u(f)) = \hat{I}_q(\varphi) < u(x_\varepsilon) = \hat{I}_q(u(x_\varepsilon))$, yielding that $f \not\succsim_Q^* x_\varepsilon$. Since $\left(\succsim_Q^*, \succsim_Q\right)$ jointly satisfy caution, we have that $x_\varepsilon \succsim_Q f$ for all $\varepsilon > 0$. By Step 5, this implies that $u(x) + \varepsilon = u(x_\varepsilon) = I_Q(u(x_\varepsilon)) \geq I_Q(u(f)) = I_Q(\varphi)$ for all $\varepsilon > 0$, that is, $\inf_{q \in Q} \hat{I}_q(\varphi) = u(x) \geq I_Q(\varphi)$, proving the step. $\square$

Step 11. For each $Q \in \mathcal{Q}$ we have that

$$f \succsim_Q g \iff \min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + \min_{q \in Q} c(p, q) \right\} \geq \min_{p \in \Delta} \left\{ \int u(g) \, \mathrm{d}p + \min_{q \in Q} c(p, q) \right\}$$

Proof of the Step Fix $Q \in \mathcal{Q}$. By Step 5, we have that

$$f \succsim_Q g \iff I_Q(u(f)) \geq I_Q(u(g))$$

By Steps 2, 9 and 10 and the definition of c, we have that

$$I_Q(u(f)) = \inf_{q\in Q} \hat{I}_q(u(f)) = \inf_{q\in Q}\inf_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} = \inf_{p\in\Delta}\inf_{q\in Q}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\}$$
$$= \inf_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + \inf_{q\in Q} c(p,q)\right\} \qquad \forall f\in\mathcal{F}$$

Since c is lower semicontinuous, we can conclude that both infima are minima and the statement follows. $\square$

Step 1 proves that u is affine and onto. Steps 2, 6, 7 and 8 prove that c is a divergence which is convex in the first argument. Steps 3 and 11 yield the representation of $\succsim_Q^*$ and $\succsim_Q$ for all $Q\in\mathcal{Q}$. As for uniqueness, cardinal uniqueness of u is routine. As for c, assume that the function $\tilde{c}:\Delta\times\Delta^\sigma\to[0,\infty]$ is a divergence which is convex in the first argument and represents $\succsim_Q^*$ and $\succsim_Q$ for all $Q\in\mathcal{Q}$. By Proposition 6 of Maccheroni et al. (2006) and since $\operatorname{Im} u=\mathbb{R}$ and $\succsim_q^*$ is a variational preference for all $q\in\Delta^\sigma$, it follows that $\tilde{c}(\cdot,q)=c(\cdot,q)$ for all $q\in\Delta^\sigma$, yielding that $c=\tilde{c}$. ■

Proof of Proposition 7 We only prove (i) implies (ii), the converse being routine. We keep the same notation and terminology of the statement and proof of Theorem 1. It is then immediate to note that Steps 1–9 of that proof continue to hold here.[40] In particular, there exist an onto and affine function u and a divergence $c:\Delta\times\Delta^\sigma\to[0,\infty]$, which is convex in the first argument, such that for each $Q\in\mathcal{Q}$

$$f\succsim_Q^* g \iff \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} \geq \min_{p\in\Delta}\left\{\int u(g)\,\mathrm{d}p + c(p,q)\right\} \quad \forall q\in Q \tag{44}$$

proving (30). Moreover, for each $Q\in\mathcal{Q}$ there exists a normalized and monotone functional $I_Q: B_0(\Sigma)\to\mathbb{R}$ such that

$$f\succsim_Q g \iff I_Q(u(f)) \geq I_Q(u(g)) \tag{45}$$

and for each $q\in\Delta^\sigma$

$$I_q(\varphi) = \min_{p\in\Delta}\left\{\int \varphi dp + c(p,q)\right\} \qquad \forall\varphi\in B_0(\Sigma)$$

Fix $Q\in\mathcal{Q}$. Given $\varphi\in B_0(\Sigma)$, note that the map $q\mapsto I_q(\varphi)$ is such that $\min_{s\in S}\varphi(s)\leq I_q(\varphi)\leq\max_{s\in S}\varphi(s)$ for all $q\in Q$, yielding that the map $q\mapsto I_q(\varphi)$ is an element of $B(Q)$. Consider the set

$$M=\{\tilde{\varphi}\in B(Q): \exists\varphi\in B_0(\Sigma) \text{ s.t. } \tilde{\varphi}(q)=I_q(\varphi) \quad \forall q\in Q\}$$

[40] The axiom of caution has been used only in the proof of Step 10 and, as a consequence, Step 11. Moreover, we only used Q-coherence for singletons in Steps 6 and 7.

Since $I_q(k1_S) = k$ for all $k \in \mathbb{R}$ and for all $q \in Q$, we have that M contains all the constants $k1_Q$ where $k \in \mathbb{R}$. Define $\tilde{J}_Q : M \to \mathbb{R}$ by $\tilde{J}_Q(\tilde{\varphi}) = I_Q(\varphi)$ where $\varphi \in B_0(\Sigma)$ is such that $\tilde{\varphi}(q) = I_q(\varphi)$ for all $q \in Q$. Note that for each $\varphi \in B_0(\Sigma)$ there exists $f \in \mathcal{F}$ such that $u(f) = \varphi$. Assume that given $\tilde{\varphi} \in M$ there exist $\varphi, \psi \in B_0(\Sigma)$ such that $\tilde{\varphi}(q) = I_q(\varphi) = I_q(\psi)$ for all $q \in Q$. Consider $f, g \in \mathcal{F}$ such that $u(f) = \varphi$ and $u(g) = \psi$. It follows that $I_q(u(f)) = I_q(u(g))$ for all $q \in Q$. By (44) and consistency, this implies that $f \sim_Q^* g$ and $f \sim_Q g$. By (45), it follows that $I_Q(\varphi) = I_Q(u(f)) = I_Q(u(g)) = I_Q(\psi)$, proving that $\tilde{J}_Q$ is well defined. Next, assume that $\tilde{\varphi}, \tilde{\psi} \in M$ are such that $\tilde{\varphi} \geq \tilde{\psi}$. Let $\varphi, \psi \in B_0(\Sigma)$ be such that $\tilde{\varphi}(q) = I_q(\varphi)$ and $\tilde{\psi}(q) = I_q(\psi)$ for all $q \in Q$. Consider $f, g \in \mathcal{F}$ such that $u(f) = \varphi$ and $u(g) = \psi$. It follows that $I_q(u(f)) \geq I_q(u(g))$ for all $q \in Q$. By (44) and consistency, this implies that $f \succsim_Q^* g$ and $f \succsim_Q g$. By (45), it follows that

$$\tilde{J}_Q(\tilde{\varphi}) = I_Q(\varphi) = I_Q(u(f)) \geq I_Q(u(g)) = I_Q(\psi) = \tilde{J}_Q\left(\tilde{\psi}\right)$$

proving that $\tilde{J}_Q$ is monotone. Moreover, by construction, we have $\tilde{J}_Q(k1_Q) = I_Q(k1_S) = k$ for all $k \in \mathbb{R}$, proving that $\tilde{J}_Q$ is normalized. By (45) and definition of $\tilde{J}_Q$, we can conclude that

$$f \succsim_Q g \iff \tilde{J}_Q\left(\min_{p \in \Delta}\left\{\int u(f)\, \mathrm{d}p + c(p, \cdot)\right\}\right) \geq \tilde{J}_Q\left(\min_{p \in \Delta}\left\{\int u(g)\, \mathrm{d}p + c(p, \cdot)\right\}\right) \qquad (46)$$

We next extend $\tilde{J}_Q$ to the entire set $B(Q)$. Define $J_Q : B(Q) \to \mathbb{R}$ by

$$J_Q(\tilde{\varphi}) = \sup\left\{\tilde{J}_Q\left(\tilde{\psi}\right) : M \ni \tilde{\psi} \leq \tilde{\varphi}\right\} \qquad \forall \tilde{\varphi} \in B(Q)$$

It is routine to check that J_Q extends $\tilde{J}_Q$ and is normalized and monotone. Moreover, by (46) and since it is an extension, it satisfies (31), proving the implication. Uniqueness follows from the same arguments of Theorem 1. ■

B.3 Remaining proofs

B.3.1 Misspecification attitudes

Proof of Proposition 2 (i) is equivalent to (ii). Given a robust two-preference family P_Q and $Q \in \mathcal{Q}$, the arguments leading to (23) and (24) allow us to conclude that $\succsim_Q^*$ and $\succsim_Q$ have the same uncertainty attitudes, yielding the equivalence. (ii) is equivalent to (iii). Consider $i \in \{1, 2\}$. Since c_i is a divergence, we have that $p \mapsto C_i(p, Q)$ is well defined, grounded and lower semicontinuous. By assumption, $p \mapsto C_i(p, Q)$ is convex for all $i \in \{1, 2\}$. By Propositions 6 and 8 of Maccheroni et al. (2006) and since u_1 and u_2 are onto, the equivalence follows. ■

Proof of Corollary 1 (i) is equivalent to (ii). By Proposition 2, the equivalence follows at each $Q \in \mathcal{Q}$, so does in general.

(ii) implies (iii). By Propositions 6 and 8 of Maccheroni et al. (2006) and since u_1 and u_2 are onto and c_1 and c_2 are divergences which are convex in the first argument, the implication follows.

(iii) implies (iv). Since $C_i(p,Q) = \min_{q \in Q} c_i(p,q)$ for all $p \in \Delta$, for all $Q \in \mathcal{Q}$, and for all $i \in \{1,2\}$, the implication trivially follows.

(iv) implies (ii). Fix $Q \in \mathcal{Q}$. Consider $f \succsim_{1,Q} x$. Since $C_1 \leq C_2$, this implies that

$$\min_{p \in \Delta}\left\{\int u(f)\,\mathrm{d}p + C_2(p,Q)\right\} \geq \min_{p \in \Delta}\left\{\int u(f)\,\mathrm{d}p + C_1(p,Q)\right\} \geq u(x)$$

proving that $f \succsim_{2,Q} x$ and, in particular, the implication. ∎

Before proving the next results, it will be useful to make few observations. Consider a robust two-preference family $P_{\mathcal{Q}}$ and fix $Q \in \mathcal{Q}$. By the proof of Proposition 2 (cf. (23) and (24)), recall that for each $f \in \mathcal{F}$ and for each $x \in X$

$$f \succsim_Q^* x \iff f \succsim_Q x \tag{47}$$

By Theorem 1, recall also that there exist an onto affine function $u : X \to \mathbb{R}$ and a divergence c, convex in p, such that

$$f \succsim_Q^* g \iff \min_{p \in \Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} \geq \min_{p \in \Delta}\left\{\int u(g)\,\mathrm{d}p + c(p,q)\right\} \qquad \forall q \in Q \tag{48}$$

and

$$f \succsim_Q g \iff \min_{p \in \Delta}\left\{\int u(f)\,\mathrm{d}p + \min_{q \in Q} c(p,q)\right\} \geq \min_{p \in \Delta}\left\{\int u(g)\,\mathrm{d}p + \min_{q \in Q} c(p,q)\right\} \tag{49}$$

In particular, it is easy to see that $\succsim_Q$ is axiomatically a variational preference for all $Q \in \mathcal{Q}$. By Theorem 3 and Proposition 6 of Maccheroni et al. (2006) and since each $\succsim_Q$ is axiomatically a variational preference, for each $Q \in \mathcal{Q}$ there exists a unique grounded, lower semicontinuous and convex function $d_Q : \Delta \to [0,\infty]$ such that (49) holds with d_Q in place of $C(\cdot, Q)$.[41] Moreover, by (7) of Maccheroni et al. (2006) and since $C(p,Q) = 0$ for all $p \in Q$, we have that $d_Q \leq C(\cdot,Q) \leq \delta_Q$. By Lemma 4 and since each $\succsim_Q$ satisfies subjective Q-coherence and since $\succsim_Q$ and $\succsim_Q^*$ coincide on X, we have that $d_Q^{-1}(0) = \overline{\text{co}}Q$.

Proof of Proposition 3 (ii) implies (i). It is trivial. (i) implies (ii). We prove the implication by only assuming that $P_{\mathcal{Q}}$ is sensitive. By Proposition 7 and since $P_{\mathcal{Q}}$ is sensitive, we have that there exist an onto affine $u : X \to \mathbb{R}$ and a divergence $c : \Delta \times \Delta^\sigma \to [0,\infty]$, convex in p, such

[41] Recall that $p \mapsto C(p,Q)$ might not be convex, yielding that a priori $d_Q \neq C(\cdot,Q)$.

that

$$f \succsim_Q^* g \Longleftrightarrow \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} \geq \min_{p\in\Delta}\left\{\int u(g)\,\mathrm{d}p + c(p,q)\right\} \qquad \forall q \in Q \tag{50}$$

By Theorem 1 of Gilboa et al. (2010) and since $\succsim_Q^*$ is a dominance relation and satisfies independence, we have that there exists a unique closed and convex set C of Δ such that

$$f \succsim_Q^* g \iff \int u(f)\,\mathrm{d}p \geq \int u(g)\,\mathrm{d}p \qquad \forall p \in C \tag{51}$$

Consider $f \in \mathcal{F}$. Define now $\hat{x}_f, \tilde{x}_f \in X$ by

$$u(\hat{x}_f) = \min_{p\in C}\int u(f)\,\mathrm{d}p \quad \text{and} \quad u(\tilde{x}_f) = \min_{q\in Q}\left\{\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\}\right\}$$

By (51) and (50), we have that

$$\int u(f)\,\mathrm{d}p \geq u(\hat{x}_f) \quad \forall p \in C \implies f \succsim_Q^* \hat{x}_f \implies \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} \geq u(\hat{x}_f) \quad \forall q \in Q$$
$$\implies u(\tilde{x}_f) = \min_{q\in Q}\left\{\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\}\right\} \geq u(\hat{x}_f).$$

By (50) and (51), we have that

$$\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\} \geq u(\tilde{x}_f) \quad \forall q \in Q \implies f \succsim_Q^* \tilde{x}_f \implies \int u(f)\,\mathrm{d}p \geq u(\tilde{x}_f) \quad \forall p \in C$$
$$\implies u(\hat{x}_f) = \min_{p\in C}\int u(f)\,\mathrm{d}p \geq u(\tilde{x}_f).$$

Since f was arbitrarily chosen, we can conclude that $u(\hat{x}_f) = u(\tilde{x}_f)$, that is, $\min_{p\in C}\int u(f)\,\mathrm{d}p = \min_{q\in Q}\left\{\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + c(p,q)\right\}\right\}$ for all $f \in \mathcal{F}$. Since u is onto, this implies that $B_0(\Sigma) = \{u(f) : f \in \mathcal{F}\}$ and $I_Q(\varphi) = \min_{p\in C}\int \varphi dp$ for all $\varphi \in B_0(\Sigma)$ where $I_Q : B_0(\Sigma) \to \mathbb{R}$ is defined as

$$I_Q(\varphi) = \min_{q\in Q}\left\{\min_{p\in\Delta}\left\{\int \varphi dp + c(p,q)\right\}\right\} = \min_{q\in Q} I_q(\varphi) \quad \forall \varphi \in B_0(\Sigma)$$

and $I_q(\varphi) = \min_{p\in\Delta}\left\{\int \varphi dp + c(p,q)\right\}$ for all $\varphi \in B_0(\Sigma)$ and for all $q \in Q$. By Theorem 2.4.18 in Zalinescu (2002) and since $p \mapsto c(p,q)$ is lower semicontinuous and convex in p and such that $\operatorname{argmin} c(\cdot,q) = \{q\}$ for all $q \in Q$, we have that

$$\overline{\text{co}}Q = \overline{\text{co}}\left(\cup_{q\in Q}\partial I_q(0)\right) = \overline{\text{co}}\left(\cup_{q\in Q: I_q(0)=I_Q(0)}\partial I_q(0)\right) = \partial I_Q(0) = C$$

proving (25).

Finally, by Steps 5, 9, and 10 of the proof of Theorem 1, if $P_{\mathcal{Q}}$ is robust, then $f \succsim_Q g$ if and only if $I_Q(u(f)) \geq I_Q(u(g))$. Since $I_Q(\varphi) = \min_{p \in C} \int \varphi dp$ for all $\varphi \in B_0(\Sigma)$ and $C = \overline{\text{co}}Q$, we have that $I_Q(\varphi) = \min_{p \in \overline{\text{co}}Q} \int \varphi dp = \min_{p \in Q} \int \varphi dp$ for all $\varphi \in B_0(\Sigma)$, proving (26). ■

Proof of Proposition 4 (i) is equivalent to (ii). Assume that $\succsim_Q^*$ satisfies c-independence. By (47), we have that if $f \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1]$, then

$$f \succsim_Q x \iff f \succsim_Q^* x \iff \alpha f + (1-\alpha) y \succsim_Q^* \alpha x + (1-\alpha) y \iff \alpha f + (1-\alpha) y \succsim_Q \alpha x + (1-\alpha) y$$

proving that $\succsim_Q$ satisfies c-independence. If $\succsim_Q$ were to satisfy c-independence, then the same argument, inverting the roles of $\succsim_Q$ and $\succsim_Q^*$, would yield the opposite implication.

(ii) implies (iv). By Propositions 6 and 19 of Maccheroni et al. (2006) and since u is onto, $d_Q^{-1}(0) = \overline{\text{co}}Q$, and $\succsim_Q$ satisfies c-independence, we have that $\delta_Q \geq C(\cdot, Q) \geq d_Q = \delta_{\overline{\text{co}}Q}$, proving the first part of the implication. Since Q is compact, if Q is convex, then $\overline{\text{co}}Q = Q$ and, in particular, $\delta_Q \geq C(\cdot, Q) \geq \delta_Q$, proving the second part.

(iv) implies (iii). Since $\delta_{\overline{\text{co}}Q} \leq C(\cdot, Q) \leq \delta_Q$, we have that for each $f \in \mathcal{F}$

$$\min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + \delta_{\overline{\text{co}}Q}(p) \right\} \leq \min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + C(p, Q) \right\} \leq \min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + \delta_Q(p) \right\}$$

Since for each $f \in \mathcal{F}$

$$\min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + \delta_{\overline{\text{co}}Q}(p) \right\} = \min_{p \in \overline{\text{co}}Q} \int u(f) \, \mathrm{d}p = \min_{p \in Q} \int u(f) \, \mathrm{d}p = \min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + \delta_Q(p) \right\}$$

this implies that

$$\min_{p \in \Delta} \left\{ \int u(f) \, \mathrm{d}p + C(p, Q) \right\} = \min_{q \in Q} \int u(f) \, \mathrm{d}q \quad \forall f \in \mathcal{F}$$

By (49), the implication follows.

(iii) implies (ii). It is routine.

(iv) is equivalent to (v). Since $C(p, Q) = \min_{q' \in Q} c(p, q') \leq c(p, q)$ for all $p \in \Delta$ and for all $q \in Q$, if $\delta_{\overline{\text{co}}Q} \leq C(\cdot, Q) \leq \delta_Q$, then $\infty = \delta_{\overline{\text{co}}Q}(p) \leq C(p, Q) \leq c(p, q)$ for all $p \notin \overline{\text{co}}Q$ and for all $q \in Q$. Vice versa, since $C(p, Q) = \min_{q' \in Q} c(p, q')$ for all $p \in \Delta$, if $c(p, q) = \infty$ for all $p \notin \overline{\text{co}}Q$ and for all $q \in Q$, then $C(p, Q) = \infty = \delta_{\overline{\text{co}}Q}(p)$ for all $p \notin \overline{\text{co}}Q$. Since $0 \leq C(\cdot, Q) \leq \delta_Q$, this implies that $\delta_{\overline{\text{co}}Q} \leq C(\cdot, Q) \leq \delta_Q$. ■

Proof of Theorem 2 We prove the "only if", the converse being obvious. Consider d_Q as defined above. Define $\gtrsim_Q^*$ by $f \gtrsim_Q^* g$ if and only if $\int u(f) \, \mathrm{d}q \geq \int u(g) \, \mathrm{d}q$ for all $q \in \overline{\text{co}}Q$. By hypothesis, the pair $\left(\gtrsim_Q^*, \succsim_Q\right)$ satisfies consistency.[42] Let $f \not\gtrsim_Q^* x$. Then, there exists $q \in \overline{\text{co}}Q$

[42]It is immediate to verify that $f \gtrsim_Q^* g$ if and only if $\int u(f) \, dq \geq \int u(g) \, dq$ for all $q \in Q$.

such that $u(x_f^q) = \int u(f)\,\mathrm{d}q < u(x)$. Hence, $x \succ_Q x_f^q$. By Lemma 4 and since $d_Q^{-1}(0) = \overline{\text{co}}Q$, we have that $x \succ_Q f$. So, the pair $\left(\gtrsim_Q^*, \succsim_Q\right)$ satisfies default to certainty. By Theorem 4 of Gilboa et al. (2010), this pair admits the representation

$$f \gtrsim_Q^* g \Longleftrightarrow \int u(f)\,\mathrm{d}q \geq \int u(g)\,\mathrm{d}q \quad \forall q \in \overline{\text{co}}Q$$

and

$$f \succsim_Q g \Longleftrightarrow \min_{q \in \overline{\text{co}}Q} \int u(f)\,\mathrm{d}q \geq \min_{q \in \overline{\text{co}}Q} \int u(g)\,\mathrm{d}q$$

Note that, in the notation of Gilboa et al. (2010), we have $C = \overline{\text{co}}Q$ because C is unique up to closure and convexity and $\overline{\text{co}}Q$ is closed and convex. Since $\min_{q \in Q} \int u(f)\,\mathrm{d}q = \min_{q \in \overline{\text{co}}Q} \int u(f)\,\mathrm{d}q$ for all $f \in \mathcal{F}$, the statement follows. ■

Proof of Corollary 2 (i) implies (ii). Fix $Q \in \mathcal{Q}$. By Proposition 3 and if $\succsim_Q^*$ is misspecification neutral at Q, then

$$f \succsim_Q^* g \iff \int u(f)\,\mathrm{d}q \geq \int u(g)\,\mathrm{d}q \quad \forall q \in Q$$

Since $\left(\succsim_Q^*, \succsim_Q\right)$ jointly satisfy consistency, $\succsim_Q$ is misspecification neutral at Q.

(ii) implies (iii). Consider $q \in \Delta^\sigma$. By Theorem 2 and since $\succsim_q$ is misspecification neutral, $f \succsim_q g$ if and only if $\int u(f)\,\mathrm{d}q \geq \int u(g)\,\mathrm{d}q$. In other words, $\succsim_q$ is represented by the functional $V_q : \mathcal{F} \to \mathbb{R}$ defined by

$$V_q(f) = \min_{p \in \Delta} \left\{ \int u(f)\,\mathrm{d}p + \delta_{\{q\}}(p) \right\} \qquad \forall f \in \mathcal{F}$$

By Proposition 6 of Maccheroni et al. (2006) and since $p \mapsto c(p,q)$ is grounded, lower semicontinuous and convex and u is onto, we have that $c(\cdot, q) = \delta_q$, proving the implication.

(iii) implies (i). Fix $Q \in \mathcal{Q}$. By (48) and since $c(p,q) = \delta_{\{q\}}(p)$ for all $p \in \Delta$ and for all $q \in \Delta^\sigma$, it follows that $f \succsim_Q^* g$ if and only if $\int u(f)\,\mathrm{d}q \geq \int u(g)\,\mathrm{d}q$ for all $q \in Q$, proving that $\succsim_Q^*$ satisfies independence and, in particular, is misspecification neutral at Q.

Finally, (27) is proved in (iii) implies (i) while (28) follows from point (ii) paired with Theorem 2. ■

Proof of Proposition 5 Consider first $\lambda \in (0, \infty)$. By Lemma 15 of Maccheroni et al. (2006), $c(\cdot, q) = \lambda D_\phi(\cdot || q)$ is Shur convex (with respect to q) for all $q \in Q$. Consider $A, B \in \Sigma$. Assume that $q(A) \geq q(B)$ for all $q \in Q$. Let $q \in Q$. Consider $x, y \in X$ such that $x \succ_Q y$. It follows that

$$\int v(u(xAy))\,\mathrm{d}q \geq \int v(u(xBy))\,\mathrm{d}q$$

for each $v : \mathbb{R} \to \mathbb{R}$ increasing and concave. By Theorem 2 of Cerreia-Vioglio et al. (2012) and since q was arbitrarily chosen, it follows that

$$\min_{p\in\Delta}\left\{\int u(xAy)\,\mathrm{d}p+\lambda D_\phi(p||q)\right\}\geq\min_{p\in\Delta}\left\{\int u(xBy)\,\mathrm{d}p+\lambda D_\phi(p||q)\right\}\quad\forall q\in Q$$

yielding that $xAy \succsim_Q^* xBy$ and, in particular, $xAy \succsim_Q xBy$. If $\lambda=\infty$ instead, we have that $c(\cdot,q)=\lambda D_\phi(\cdot||q)=\delta_{\{q\}}(\cdot)$ for all $q\in Q$. This implies that (18) takes the max-min form over the set Q, which trivially implies bet-consistency. ■

Functional approach for $\succsim_Q$ In the Introduction we outlined a "protective belt" interpretation of decision criterion

$$V_Q(f)=\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p+\min_{q\in Q}c(p,q)\right\}$$

In Proposition 2 we observed that $p\mapsto C(p,Q)=\min_{q\in Q}c(p,q)$ is an index of misspecification aversion: the higher the fear, the lower the index. This misspecification index has the following bounds

$$0\leq\min_{q\in Q}c(p,q)\leq\delta_Q(p)\qquad\forall p\in\Delta\tag{52}$$

The upper bound δ_Q suggests that fear of misspecification is absent when the misspecification index is δ_Q – e.g., when $\lambda=+\infty$ in (19) – in which case criterion (18) takes a Wald (1950) max-min form

$$V_Q(f)=\min_{q\in Q}\int u(f)\,\mathrm{d}q\tag{53}$$

This max-min criterion characterizes a decision maker who confronts model misspecification, but is not concerned by it and exhibits only aversion to model ambiguity. In other words, this Waldean decision maker is again a natural candidate to be (model) misspecification neutral for $\succsim_Q$. The next limit result further corroborates this insight by showing that, when the fear of misspecification vanishes, the decision maker becomes Waldean.[43]

Proposition 10 *If Q is compact, then for each $f\in\mathcal{F}$,*

$$\lim_{\lambda\uparrow\infty}\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p+\lambda\min_{q\in Q}R(p||q)\right\}=\min_{q\in Q}\int u(f)\,\mathrm{d}q$$

Proof First, note that $\min_{q\in Q}R(p||q)=0$ if and only if $p\in Q$. Indeed, we have that

$$\min_{q\in Q}R(p||q)=0\iff\exists\bar{q}\in Q\text{ s.t. }R(p||\bar{q})=0\iff\exists\bar{q}\in Q\text{ s.t. }p=\bar{q}$$

[43] To ease matters, we state the result in terms of criterion (19). A general version can be easily established via an increasing sequence of misspecification indexes.

Define $\lambda_n = n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $\lambda_n \min_{q\in Q} R(p||q) = 0$ if and only if $p \in Q$. So, for each $p \in \Delta$,

$$\lim_n \lambda_n \min_{q\in Q} R(p||q) = \begin{cases} 0 & \text{if } p \in Q \\ +\infty & \text{if } p \notin Q \end{cases}$$

Since $\lambda_n \min_{q\in Q} R(p||q) = 0$ for each $n \in \mathbb{N}$ if and only if $p \in Q$, by Proposition 5.4, Remark 5.5, and Theorem 7.4 of Dal Maso (1993) we have

$$\lim_n \min_{p\in\Delta} \left\{ \int u(f)\,\mathrm{d}p + \lambda_n \min_{q\in Q} R(p||q) \right\} = \min_{q\in Q} \int u(f)\,\mathrm{d}q \quad \forall f \in \mathcal{F}$$

Finally, by (52), we have that for each $f \in \mathcal{F}$

$$\begin{aligned} \min_{q\in Q} \int u(f)\,\mathrm{d}q &\le \lim_n \min_{p\in\Delta} \left\{ \int u(f)\,\mathrm{d}p + \lambda_n \min_{q\in Q} R(p||q) \right\} \\ &\le \lim_{\lambda\uparrow\infty} \min_{p\in\Delta} \left\{ \int u(f)\,\mathrm{d}p + \lambda \min_{q\in Q} R(p||q) \right\} \le \min_{q\in Q} \int u(f)\,\mathrm{d}q \end{aligned}$$

yielding the statement. ∎

B.3.2 Remaining results

The proof of Proposition 1 follows immediately from the following lemma. Here, as usual, ϕ is extended to $\mathbb{R}$ by setting $\phi(t) = +\infty$ if $t \notin [0,\infty)$. In particular, ϕ^* is real valued and increasing.

Lemma 6 *For each $Q \subseteq \Delta^\sigma$ and for each $\lambda \in (0,\infty)$,*

$$\inf_{p\in\Delta} \left\{ \int u(f)\,\mathrm{d}p + \lambda \inf_{q\in Q} D_\phi(p||q) \right\} = \lambda \inf_{q\in Q} \sup_{\eta\in\mathbb{R}} \left\{ \eta - \int \phi^*\left(\eta - \frac{u(f)}{\lambda}\right) \mathrm{d}q \right\}$$

for all $u : X \to \mathbb{R}$ and all $f : S \to X$ such that $u \circ f$ is bounded and Σ-measurable.

Proof By Theorem 4.2 of Ben-Tal and Teboulle (2007), for each $q \in \Delta^\sigma$ it holds

$$\inf_{p\in\Delta} \left\{ \int \xi dp + D_\phi(p||q) \right\} = \sup_{\eta\in\mathbb{R}} \left\{ \eta - \int \phi^*(\eta - \xi)\,\mathrm{d}q \right\}$$

for all $\xi \in L^{\infty}(q)$. Then, if $u \circ f$ is bounded and measurable, then $u \circ f \in L^{\infty}(q)$ for all $q \in \Delta^{\sigma}$, it follows that

$$\inf_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda D_{\phi}(p||q) \right\} = \lambda \inf_{p \in \Delta} \left\{ \int \frac{u(f)}{\lambda} \mathrm{d}p + D_{\phi}(p||q) \right\}$$
$$= \lambda \sup_{\eta \in \mathbb{R}} \left\{ \eta - \int \phi^{*} \left(\eta - \frac{u(f)}{\lambda} \right) \mathrm{d}q \right\}$$

for all $\lambda > 0$, as desired. By taking the inf over Q on both sides of the equation, the statement follows. ■

Proof of Proposition 6 We only prove (i) implies (ii), the converse and uniqueness being routine. We keep the notation of the proof of Theorem 1. Compared to that result, we only need to prove that c is jointly convex. Fix $\varphi \in B_0(\Sigma)$, $q, q' \in \Delta^{\sigma}$ and $\lambda \in (0,1)$. By model hybridization aversion and since u is affine, we have that

$$J(\varphi, \lambda q + (1-\lambda) q') = u\left(x_{f,\lambda q + (1-\lambda) q'}\right) \leq u\left(\lambda x_{f,q} + (1-\lambda) x_{f,q'}\right)$$
$$= \lambda u(x_{f,q}) + (1-\lambda) u(x_{f,q'}) = \lambda J(\varphi, q) + (1-\lambda) J(\varphi, q')$$

where $f \in \mathcal{F}$ is such that $u(f) = \varphi$. Since φ, q, q' and λ were arbitrarily chosen, this yields that J is convex in the second argument. Since J is convex in the second argument, the map $(p, q) \mapsto J(\varphi, q) - \int \varphi dp$, defined over $\Delta \times \Delta^{\sigma}$, is jointly convex for all $\varphi \in B_0(\Sigma)$. By (43) and the definition of c, we conclude that c is convex, proving the implication. ■

Proof of Corollary 3 (i) implies (ii). By the definitions of robust and sensitive, it is immediate. (ii) implies (iii). A careful inspection of the proof of (i) implies (ii) in Theorem 1 reveals that we only used Q-coherence restricted to singletons in Steps 6 and 7, proving the implication. (iii) implies (i). It is implication (ii) implies (i) of Theorem 1. As for uniqueness, given the equivalence between (i) and (iii), it follows again from Theorem 1. ■

Proof of Proposition 8 We begin by making two observations. It is well known that, given a continuous function $F : Q \to \mathbb{R}$,

$$\lim_{\xi \to 0^{+}} \phi_{\xi}^{-1} \left(\int_{Q} \phi_{\xi}(F(q)) \, d\mu_{Q}(q) \right) = \min_{q \in \operatorname{supp} \mu_{Q}} F(q) = \min_{q \in Q} F(q) \tag{54}$$

and

$$\phi_{\xi}^{-1} \left(\int_{Q} \phi_{\xi}(F(q)) \, d\mu_{Q} \right) = \min_{\nu \ll \mu_{Q}} \left\{ \int F d\nu + \xi R(\nu||\mu_{Q}) \right\} \tag{55}$$

where $\phi_{\xi}(t) = -e^{-\frac{1}{\xi}t}$ for all $t \in \mathbb{R}$ and $\xi > 0$. Fix $f \in \mathcal{F}$ and $\lambda \in (0, \infty]$. If $\lambda < \infty$, then set $F_{\lambda}(q) = \min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda R(p||q) \right\} = \phi_{\lambda}^{-1} \left(\int \phi_{\lambda}(u(f)) \,\mathrm{d}q \right)$ for all $q \in Q$, where $\phi_{\lambda}(t) = -e^{-\frac{1}{\lambda}t}$ for all $t \in \mathbb{R}$. If $\lambda = \infty$, then set $F_{\lambda}(q) = \int u(f) \,\mathrm{d}q$ for all $q \in Q$. Since each $f \in \mathcal{F}$ is finitely valued, it is immediate to see that in both cases F_{λ} is continuous.

By (54), (34) follows. By Proposition 12 of Maccheroni et al. (2006) and (55) and since $\lim_{\xi\to\infty}\xi R(\nu||\mu_Q)=\infty$ if $\nu\neq\mu_Q$ and $\lim_{\xi\to\infty}\xi R(\nu||\mu_Q)=0$ if $\nu=\mu_Q$, (35) follows. By (35), we have that

$$\lim_{\xi\to\infty} V_Q^{\lambda,\xi}(f)=\int_Q\left(\min_{p\in\Delta}\left\{\int_S u(f(s))\,\mathrm{d}p(s)+\lambda R(p||q)\right\}\right)d\mu_Q(q)$$

By Proposition 12 of Maccheroni et al. (2006) and since $\lim_{\lambda\to\infty}\lambda R(p||q)=\infty$ if $p\neq q$ and $\lim_{\lambda\to\infty}\lambda R(p||q)=0$ if $p=q$, we have that $\lim_{\lambda\to\infty}F_\lambda(q)=\int u(f)\,\mathrm{d}q=F_\infty(q)$ for all $q\in Q$. By the Lebesgue Dominated Convergence Theorem (applied to any sequence in $\{F_\lambda\}_{\lambda\in(0,\infty)}$) and since $\{F_\lambda\}_{\lambda\in(0,\infty)}$ is uniformly bounded, the second equality of (36) follows. The first has a similar proof and we omit it. ■

B.3.3 Appendix A

Proof of Lemma 1 Set $T=\operatorname{int}\operatorname{Im}u$. Since u is non-constant, T is a non-empty interval. Without loss of generality, we assume that $0,1\in T$. Otherwise, it is enough to replace u with $\tilde{u}=au+b$, where $a,b\in\mathbb{R}$ and $a>0$ are such that $0,1\in\operatorname{int}\operatorname{Im}\tilde{u}$. Accordingly, we replace ϕ with $\tilde{\phi}:\operatorname{Im}\tilde{u}\to\mathbb{R}$ defined by $\tilde{\phi}(t)=\phi\left(\frac{t-b}{a}\right)$ for all $t\in\operatorname{Im}\tilde{u}$. These changes and transformations yield ordinally the same V (it becomes $aV+b$) and leave the properties of u and ϕ unchanged, but with $0,1\in\operatorname{int}\operatorname{Im}\tilde{u}$.

1. We prove the “only if”, the converse being obvious. Since $\succsim$ satisfies convexity, it has convex upper contour sets, i.e., given any $f,g,h\in\mathcal{F}$, if $f\succsim h$ and $g\succsim h$, then $\gamma f+(1-\gamma)g\succsim h$ for all $\gamma\in(0,1)$. Consider an essential event E and define $\alpha=\min_{q'\in Q'}q'(E)\in(0,1)$. Define $F:T\times T\to\mathbb{R}$ by, for each $(t,s)\in T\times T$,

$$F(t,s)=\phi^{-1}(\alpha\phi(t)+(1-\alpha)\phi(s))$$

The set $D_{\leq}=\{(t,s)\in T\times T:t\geq s\}$ is convex and has a non-empty interior. Consider $(t,s),(t',s')\in D_{\leq}$ and $\gamma\in(0,1)$. Consider also $x,x',y,y'\in X$ such that $u(x)=t$, $u(x')=t'$, $u(y)=s$, and $u(y')=s'$. Since $(t,s),(t',s')\in D_{\leq}$ and $\succsim$ is represented by V, defined as in (39), we have $V(xEy)=F(t,s)$ and $V(x'Ey')=F(t',s')$. Similarly, we have that $F(\gamma t+(1-\gamma)t',\gamma s+(1-\gamma)s')=V(x''Ey'')$ where $x''=\gamma x+(1-\gamma)x'$ and $y''=\gamma y+(1-\gamma)y'$. Since $\succsim$ has convex upper contour sets and V represents it, we conclude that $F(\gamma t+(1-\gamma)t',\gamma s+(1-\gamma)s')\geq\min\{F(t,s),F(t',s')\}$, proving that F is quasiconcave on $D_{\leq}$. Since $D_{\leq}$ has non-empty interior, there exist two open subintervals $T',T''\subseteq T$ such that $T'\times T''\subseteq D_{\leq}$. Since ϕ is strictly increasing and F is quasiconcave on $D_{\leq}$, we have that F is quasiconcave on $T'\times T''$ and so is $\phi\circ F$. By Theorem 2 of Debreu and Koopmans (1982), we conclude that ϕ is concave.[44]

[44] In the language of Debreu and Koopmans, set $X=T'$, $Y=T''$, $f=-\alpha\phi$ and $g=-(1-\alpha)\phi$. By their

2. Also for this point we prove the "only if", the converse being obvious. Define $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ by $I(\varphi) = \min_{q' \in Q'} \phi^{-1}\left(\int \phi(\varphi)\, dq'\right)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Since ϕ is strictly increasing and continuous, it is obvious that I is normalized and monotone. Since $\succsim$ is represented by V, defined as in (39), it follows that $V = I \circ u$ represents $\succsim$. By Theorem 4 of Cerreia-Vioglio et al. (2014) and since $\succsim$ satisfies weak c-independence, I is translation invariant. Define also

$$T_r = \{t \in T : t + r \in T\} = T \cap (T - r)$$

Since T is open, T_r is an open subinterval of $T \subseteq \operatorname{Im} u$ and $0 \in T_r$ for all $r \in T$. Define $\phi_0 : T_r \to \mathbb{R}$ and $\phi_r : T_r \to \mathbb{R}$ by $\phi_0(t) = \phi(t)$ and $\phi_r(t) = \phi(t + r)$ for all $t \in T_r$. By definition of I and because it is translation invariant, we have, for each $(t, s) \in D_{\leq} \cap T_r \times T_r$,

$$F(t + r, s + r) = I((t + r) 1_E + (s + r) 1_{E_c}) = I(t 1_E + s 1_{E_c}) + r = F(t, s) + r$$

that is,

$$\begin{aligned} \phi_r^{-1}(\alpha\phi_r(t) + (1 - \alpha)\phi_r(s)) &= \phi^{-1}(\alpha\phi(t + r) + (1 - \alpha)\phi(s + r)) - r \\ &= \phi^{-1}(\alpha\phi(t) + (1 - \alpha)\phi(s)) \end{aligned}$$

By Lemma Al.l in Wakker (1989) and since ϕ is strictly increasing and continuous and r was arbitrarily chosen we conclude that, for each $r \in T$, there exist $\mu_r, \lambda_r \in \mathbb{R}$ with $\mu_r > 0$ such that $\phi_r(t) = \mu_r \phi(t) + \lambda_r$ for all $t \in T_r$. Define $\mu : T \to (0, \infty)$ and $\lambda : T \to \mathbb{R}$ by $\lambda(r) = \lambda_r$ and $\mu(r) = \mu_r$. We have that $\phi(t + r) = \mu(r)\phi(t) + \lambda(r)$ for all $t \in T_r$ and all $r \in T$, i.e., there exist $\lambda, \mu : T \to \mathbb{R}$ such that, for all $t, r \in T$ with $t + r \in T$,

$$\phi(t + r) = \mu(r)\phi(t) + \lambda(r)$$

By p. 3233 of Aczel (2005), it follows that either $\phi(r) = \alpha r + \beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ or $\phi(r) = \alpha e^{\gamma r} + \beta$ for some $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha\gamma > 0$. ■

Proof of Lemma 2 Define $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ by $I(\varphi) = \min_{q' \in Q'} \phi^{-1}\left(\int \phi(\varphi)\, dq'\right)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. Call $k \in \operatorname{int} \operatorname{Im} u$. By construction and by points 1 and 2 of Lemma 1, I is normalized, monotone, quasiconcave, and translation invariant. In particular, it is concave and so is ϕ which is also CARA. Being represented by $V = I \circ u$, $\succsim$ is thus a variational preference. Define $J : B_0(\Sigma, \operatorname{Im} u) \times Q' \to \mathbb{R}$ by $J(\varphi, q') = \phi^{-1}\left(\int \phi(\varphi)\, dq'\right)$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$ and all $q' \in Q'$. It is well known that $J(\cdot, q')$ is normalized, monotone, translation invariant, concave, and such that $\partial J(\cdot, q')(k) = \{q'\}$ for all $q' \in Q'$. Clearly, $I(\varphi) = \min_{q' \in Q'} J(\varphi, q')$ for all $\varphi \in B_0(\Sigma, \operatorname{Im} u)$. By Theorem 2.4.18 in Zalinescu (2002) and since $q' \mapsto J(\cdot, q')$ is lower

Theorem 2, it follows that either f or g is convex. Either way, since $\alpha \in (0, 1)$, ϕ is concave.

semicontinuous, we have that

$$\overline{\mathrm{co}}Q' = \overline{\mathrm{co}}\left(\cup_{q'\in Q'}\partial J\left(\cdot, q'\right)(k)\right) = \overline{\mathrm{co}}\left(\cup_{q'\in Q': J(k,q')=I(k)}\partial J\left(\cdot, q'\right)(k)\right) = \partial I\left(k\right) \tag{56}$$

We now prove the "only if". By Jensen's inequality and since ϕ is concave, we have that $x \succ x_f^{q'}$ implies $x \succ f$ for all $q' \in Q'$. By single-preference subjective Q-coherence and since V represents $\succsim$, we conclude that $q' \in \overline{\mathrm{co}}Q$ for all $q' \in Q'$, i.e., $Q' \subseteq \overline{\mathrm{co}}Q$ and $\overline{\mathrm{co}}Q' \subseteq \overline{\mathrm{co}}Q$. As for the opposite inclusion, by single-preference subjective Q-coherence, if $p \in \overline{\mathrm{co}}Q$, then for each $f \in \mathcal{F}$ and for each $x \in X$

$$x \succ x_f^p \implies x \succ f$$

By Lemma 4 and its proof, this means that $x_f^p \succsim f$ for all $f \in \mathcal{F}$, i.e., $V(f) \leq \int u(f)\,\mathrm{d}p$ for all $f \in \mathcal{F}$. So, p corresponds to a SEU preference that is less ambiguity averse than $\succsim$. By Lemma 32 of Maccheroni et al. (2006), we conclude that $p \in \partial I(k) = \overline{\mathrm{co}}Q'$, proving that $\overline{\mathrm{co}}Q' \supseteq \overline{\mathrm{co}}Q$. As for the "if", by (56) and Lemma 32 of Maccheroni et al. (2006), we have

$$p \in \Delta \text{ is s.t. } x_f^p \succsim f \quad \forall f \in \mathcal{F} \iff p \in \partial I(k) = \overline{\mathrm{co}}Q' = \overline{\mathrm{co}}Q$$

In words, p satisfies condition (ii) of Lemma 4 if and only if $p \in \overline{\mathrm{co}}Q$. By Lemma 4 and its proof, this implies that p satisfies condition (iii) of Lemma 4 if and only if p satisfies condition (ii) of Lemma 4 if and only if $p \in \overline{\mathrm{co}}Q$, proving that $\succsim$ satisfies single-preference subjective Q-coherence. ■

Proof of Lemma 3 As the converse is trivial, we only prove that (i) implies (ii). By the same techniques of Proposition 3 of Ghirardato et al. (2003), there exist a non-constant continuous function $v : X \to \mathbb{R}$ and a non-empty compact subset Q' of Δ such that $\succsim$ is represented by $\tilde{V} : \mathcal{F} \to \mathbb{R}$ defined by

$$\tilde{V}(f) = \min_{q'\in Q'} \int v(f)\,\mathrm{d}q' \tag{57}$$

where $0 < \min_{q'\in Q'} q'(E) < 1$ for some $E \in \Sigma$. By usual separation arguments, $Q'' = \overline{\mathrm{co}}Q'$ is the only non-empty compact and convex subset of Δ for which (57) holds. Chateauneuf et al. (2005) show that monotone continuity guarantees that Q' consists of (countably additive) probability measures. Since $\succsim$ is a continuous and nontrivial rational preference, there is a non-constant affine $u : X \to \mathbb{R}$ that represents $\succsim$ on X. Since v also represents $\succsim$ on X, we conclude that $v = \phi \circ u$ where $\phi : \mathrm{Im}\, u \to \mathbb{R}$ is strictly increasing and continuous. This proves that $u = \phi^{-1} \circ v$ is continuous. It follows that $V(f) = \min_{q'\in Q'} \int \phi(u(f))\,\mathrm{d}q'$ for all $f \in \mathcal{F}$. Since ϕ is strictly increasing and continuous, the functional $V : \mathcal{F} \to \mathbb{R}$ defined by

$$V(f) = \phi^{-1}\left(\min_{q'\in Q'} \int \phi(u(f))\,\mathrm{d}q'\right) = \min_{q'\in Q'} \phi^{-1}\left(\int \phi(u(f))\,\mathrm{d}q'\right)$$

represents $\succsim$, proving the implication. ■

Proof of Theorem 3 As the converse is trivial, we only prove that (i) implies (ii). Lemma 3, along with Lemmas 1 and 2, imply that the functional $V : \mathcal{F} \to \mathbb{R}$ defined by

$$V(f) = \min_{q \in Q} \phi^{-1} \left(\int \phi(u(f)) \,\mathrm{d}q \right)$$

represents $\succsim$ and $\phi : \operatorname{Im} u \to \mathbb{R}$ is strictly increasing, continuous, CARA, and concave. In other words, for each $t \in \operatorname{Im} u$ we have either $\phi(t) = at + b$ with $a > 0$ and $b \in \mathbb{R}$ or $\phi(t) = -ae^{-\lambda t} + b$ where $a, \lambda > 0$ and $b \in \mathbb{R}$. By the results of Section B.4.1 (below), we conclude that

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda \min_{q \in Q} R(p||q) \right\}$$

represents $\succsim$, as desired (when ϕ is linear, this corresponds to $\lambda = \infty$). Finally, points 1–4 are routine. ■

B.4 Additional material

B.4.1 Non-convex set of structured models

Let us consider two decision makers who adopt criterion (19), the first one posits a, possibly non-convex but compact, set of structured models Q and the second one posits its closed convex hull $\overline{\operatorname{co}}\, Q$. So, the second decision maker considers also all the mixtures of structured models posited by the first decision maker. Next we show that their preferences over acts actually agree. We deal with the case $\lambda \in (0, \infty)$, being $\lambda = \infty$ trivial. It is thus without loss of generality to assume that the set of posited structured models is convex for our entropic specification. Before doing so we prove formula (20). Observe that given a compact subset $Q \subseteq \Delta^\sigma$, be that convex or not, we have

$$\begin{aligned}
\min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda \min_{q \in Q} R(p||q) \right\} &= \min_{p \in \Delta} \min_{q \in Q} \left\{ \int u(f) \,\mathrm{d}p + \lambda R(p||q) \right\} \\
&= \min_{q \in Q} \min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda R(p||q) \right\} \\
&= \min_{q \in Q} \phi_\lambda^{-1} \left(\int \phi_\lambda(u(f)) \,\mathrm{d}q \right)
\end{aligned}$$

where $\phi_\lambda(t) = -e^{-\frac{1}{\lambda}t}$ for all $t \in \mathbb{R}$ and $\lambda > 0$. Observe that the next result, as the equalities above, does not rely on the unboundedness of u.

Proposition 11 *If $Q \subseteq \Delta^\sigma$ is compact, then for each $f \in \mathcal{F}$*

$$\min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda \min_{q \in Q} R(p||q) \right\} = \min_{p \in \Delta} \left\{ \int u(f) \,\mathrm{d}p + \lambda \min_{q \in \overline{\operatorname{co}}\, Q} R(p||q) \right\}$$

Proof First observe that $\overline{\text{co}}\, Q \subseteq \Delta^{\sigma}$. Indeed, since Q is a compact subset of Δ^{σ}, the set function $\nu : \Sigma \to [0,1]$, defined by $\nu(E) = \min_{q\in Q} q(E)$ for all $E \in \Sigma$ is an exact capacity which is continuous at S. This implies that $Q \subseteq \text{core}\,\nu \subseteq \Delta^{\sigma}$, yielding that $\overline{\text{co}}\, Q \subseteq \text{core}\,\nu \subseteq \Delta^{\sigma}$. Given what we have shown before we can conclude that

$$
\begin{aligned}
\min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + \lambda \min_{q\in Q} R(p\|q)\right\} &= \min_{q\in Q} \phi_{\lambda}^{-1}\left(\int \phi_{\lambda}(u(f))\,\mathrm{d}q\right) \\
&= \phi_{\lambda}^{-1}\left(\min_{q\in Q}\int \phi_{\lambda}(u(f))\,\mathrm{d}q\right) \\
&= \phi_{\lambda}^{-1}\left(\min_{q\in\overline{\text{co}}Q}\left(\int \phi_{\lambda}(u(f))\,\mathrm{d}q\right)\right) \\
&= \min_{q\in\overline{\text{co}}Q} \phi_{\lambda}^{-1}\left(\int \phi_{\lambda}(u(f))\,\mathrm{d}q\right) \\
&= \min_{p\in\Delta}\left\{\int u(f)\,\mathrm{d}p + \lambda \min_{q\in\overline{\text{co}}\,Q} R(p\|q)\right\}
\end{aligned}
$$

proving the statement. ■

After (22), we claimed that the Gini criterion is a monotone version of the max-min mean-variance criterion. To be more precise, given a probability $q \in \Delta^{\sigma}$ and a weight $1/2\lambda > 0$ for the variance, the mean-variance criterion is not monotone over its entire domain, but it is normalized, translation invariant, and monotone in an area containing the constant functions (see Theorem 24 and its proof of Maccheroni et al., 2006). At the same time, the variational preference with cost function the Gini index $\lambda\chi^2(\cdot\|q)$ is monotone and coincides with the mean-variance criterion over such an area. A similar argument, *mutatis mutandis*, holds for the max-min mean-variance criterion and our formula (21). This allows us to see the corresponding variational criteria as a monotonization of the corresponding mean-variance ones.

B.4.2 Representation with fixed Q

In this appendix, we provide a foundation of our main criterion by keeping Q fixed, compact and convex. The primitive will be a pair $\left(\succsim_Q^*, \succsim_Q\right) = (\succsim^*, \succsim)$ with Q fixed where $\succsim^*$ is an unbounded dominance relation, $\succsim$ is a rational preference, both are Q-coherent and jointly satisfy caution and consistency. The proof is based on two pillars. The first step (Section B.4.2) proves that $\succsim^*$ admits a multi-variational representation which can further be refined to be parametrized by Q, the second step (Section B.4.2) shows that $\succsim$ can be represented by our main criterion, given that $\succsim$ is a cautious completion of $\succsim^*$. Given $c : \Delta \times Q \to [0,\infty]$, we say that c is variational if $p \mapsto c(p,q)$ is grounded, lower semicontinuous and convex for all $q \in Q$ and $c_Q(\cdot) = \min_{q\in Q} c(\cdot, q)$ is well defined and shares the same properties. We say that a variational c is a variational pseudo-statistical distance if $c_Q^{-1}(0) = Q$.

A Bewley-type representation The next result is a multi-utility (variational) representation for unbounded dominance relations.

Lemma 7 *Let $\succsim^*$ be a binary relation on $\mathcal{F}$, where (S, Σ) is a standard Borel space. The following statements are equivalent:*

(i) *$\succsim^*$ is an unbounded dominance relation which satisfies objective Q-coherence;*

(ii) *there exist an onto affine function $u : X \to \mathbb{R}$ and a variational $c : \Delta \times Q \to [0, \infty]$ such that* $\operatorname{dom} c(\cdot, q) \subseteq \Delta^{\ll}(Q)$ *for all $q \in Q$ and*

$$f \succsim^* g \iff \min_{p \in \Delta} \left\{ \int u(f)\,\mathrm{d}p + c(p, q) \right\} \geq \min_{p \in \Delta} \left\{ \int u(g)\,\mathrm{d}p + c(p, q) \right\} \quad \forall q \in Q \tag{58}$$

To prove this result, we need to introduce one mathematical object. Let $\succeq^*$ be a binary relation on $B_0(\Sigma)$. We say that $\succeq^*$ is *convex niveloidal* if and only if $\succeq^*$ is a preorder that satisfies the following five properties:

1. For each $\varphi, \psi \in B_0(\Sigma)$ and for each $k \in \mathbb{R}$
$$\varphi \succeq^* \psi \implies \varphi + k \succeq^* \psi + k$$
2. If $\varphi, \psi \in B_0(\Sigma)$ and $\{k_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ are such that $k_n \uparrow k$ and $\varphi - k_n \succeq^* \psi$ for all $n \in \mathbb{N}$, then $\varphi - k \succeq^* \psi$;
3. For each $\varphi, \psi \in B_0(\Sigma)$,
$$\varphi \geq \psi \implies \varphi \succeq^* \psi$$
4. For each $k, h \in \mathbb{R}$ and for each $\varphi \in B_0(\Sigma)$,
$$k > h \implies \varphi + k \succ^* \varphi + h$$
5. For each $\varphi, \psi, \xi \in B_0(\Sigma)$ and for each $\lambda \in (0, 1)$,
$$\varphi \succeq^* \xi \text{ and } \psi \succeq^* \xi \implies \lambda\varphi + (1 - \lambda)\psi \succeq^* \xi$$

Lemma 8 *If $\succsim^*$ is an unbounded dominance relation, then there exists an onto affine function $u : X \to \mathbb{R}$ such that*

$$x \succsim^* y \iff u(x) \geq u(y) \tag{59}$$

Proof Since $\succsim^*$ is a non-trivial preorder on $\mathcal{F}$ that satisfies c-completeness, continuity and weak c-independence, it is immediate to conclude that $\succsim^*$ restricted to X satisfies weak order, continuity and risk independence. By Herstein and Milnor (1953), it follows that there exists an affine function $u : X \to \mathbb{R}$ that satisfies (59). Since $\succsim^*$ is a non-trivial c-complete preorder on $\mathcal{F}$ that satisfies monotonicity, we have that $\succsim^*$ is non-trivial on X. By Lemma 59 of Cerreia-Vioglio et al. (2011b) and since $\succsim^*$ is non-trivial on X and satisfies unboundedness, we can conclude that u is onto. ■

Since u is affine and onto, note that $\{u(f) : f \in \mathcal{F}\} = B_0(\Sigma)$. In light of this observation, we can define a binary relation $\succeq^*$ on $B_0(\Sigma)$ by

$$\varphi \succeq^* \psi \iff f \succsim^* g \text{ where } u(f) = \varphi \text{ and } u(g) = \psi \tag{60}$$

Lemma 9 *If $\succsim^*$ is an unbounded dominance relation, then $\succeq^*$, defined as in (60), is a well defined convex niveloidal binary relation. Moreover, if $\succsim^*$ is objectively Q-coherent, then $\varphi \overset{Q}{=} \psi$ implies $\varphi \sim^* \psi$.*

Proof We begin by showing that $\succeq^*$ is well defined and does not depend on the representing elements of ψ and φ. Assume that $f_1, f_2, g_1, g_2 \in \mathcal{F}$ are such that $u(f_i) = \varphi$ and $u(g_i) = \psi$ for all $i \in \{1, 2\}$. It follows that $u(f_1(s)) = u(f_2(s))$ and $u(g_1(s)) = u(g_2(s))$ for all $s \in S$. By Lemma 8, this implies that $f_1(s) \sim^* f_2(s)$ and $g_1(s) \sim^* g_2(s)$ for all $s \in S$. Since $\succsim^*$ is a preorder that satisfies monotonicity, this implies that $f_1 \sim^* f_2$ and $g_1 \sim^* g_2$. Since $\succsim^*$ is a preorder, if $f_1 \succsim^* g_1$, then

$$f_2 \succsim^* f_1 \succsim^* g_1 \succsim^* g_2 \implies f_2 \succsim^* g_2$$

that is, $f_1 \succsim^* g_1$ implies $f_2 \succsim^* g_2$. Similarly, we can prove that $f_2 \succsim^* g_2$ implies $f_1 \succsim^* g_1$. In other words, $f_1 \succsim^* g_1$ if and only if $f_2 \succsim^* g_2$, proving that $\succeq^*$ is well defined and does not depend on the representing elements of ψ and φ. It is immediate to prove that $\succeq^*$ is a preorder. We next prove properties 1–5.

1. Consider $\varphi, \psi \in B_0(\Sigma)$ and $k \in \mathbb{R}$. Assume that $\varphi \succeq^* \psi$. Let $f, g \in \mathcal{F}$ and $x, y \in X$ be such that $u(f) = 2\varphi$, $u(g) = 2\psi$, $u(x) = 0$ and $u(y) = 2k$. Since u is affine, it follows that

$$\begin{aligned} u\left(\frac{1}{2}f + \frac{1}{2}x\right) &= \frac{1}{2}u(f) + \frac{1}{2}u(x) = \varphi \succeq^* \psi \\ &= \frac{1}{2}u(g) + \frac{1}{2}u(x) = u\left(\frac{1}{2}g + \frac{1}{2}x\right) \end{aligned}$$

proving that $\frac{1}{2}f + \frac{1}{2}x \succsim^* \frac{1}{2}g + \frac{1}{2}x$. Since $\succsim^*$ satisfies weak c-independence and u is affine, we have that $\frac{1}{2}f + \frac{1}{2}y \succsim^* \frac{1}{2}g + \frac{1}{2}y$, yielding that

$$\begin{aligned}\varphi + k &= \frac{1}{2}u(f) + \frac{1}{2}u(y) = u\left(\frac{1}{2}f + \frac{1}{2}y\right) \succeq^* u\left(\frac{1}{2}g + \frac{1}{2}y\right) \\ &= \frac{1}{2}u(g) + \frac{1}{2}u(y) = \psi + k\end{aligned}$$

2. Consider $\varphi, \psi \in B_0(\Sigma)$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $k_n \uparrow k$ and $\varphi - k_n \succeq^* \psi$ for all $n \in \mathbb{N}$. We have two cases:

   (a) $k > 0$. Consider $f, g, h \in \mathcal{F}$ such that

   $$u(f) = \varphi,\ u(g) = \varphi - k \text{ and } u(h) = \psi$$

   Since $k > 0$ and $k_n \uparrow k$, there exists $\bar{n} \in \mathbb{N}$ such that $k_n > 0$ for all $n \geq \bar{n}$. Define $\lambda_n = 1 - k_n/k$ for all $n \in \mathbb{N}$. It follows that $\lambda_n \in [0,1]$ for all $n \geq \bar{n}$. Since u is affine, for each $n \geq \bar{n}$

   $$u(\lambda_n f + (1-\lambda_n) g) = \lambda_n u(f) + (1-\lambda_n) u(g) = \varphi - k_n \succeq^* \psi = u(h)$$

   yielding that $\lambda_n f + (1-\lambda_n) g \succsim^* h$ for all $n \geq \bar{n}$. Since $\succsim^*$ satisfies continuity and $\lambda_n \to 0$, we have that $g \succsim^* h$, that is,

   $$\varphi - k = u(g) \succeq^* u(h) = \psi$$

   (b) $k \leq 0$. Since $\{k_n\}_{n\in\mathbb{N}}$ is convergent, $\{k_n\}_{n\in\mathbb{N}}$ is bounded. Thus, there exists $h > 0$ such that $k_n + h > 0$ for all $n \in \mathbb{N}$. Moreover, $k_n + h \uparrow k + h > 0$. By point 1, we also have that $\varphi - (k_n + h) = (\varphi - k_n) - h \succeq^* \psi - h$ for all $n \in \mathbb{N}$. By subpoint a, we can conclude that $(\varphi - k) - h = \varphi - (k+h) \succeq^* \psi - h$. By point 1, we obtain that $\varphi - k \succeq^* \psi$.

3. Consider $\varphi, \psi \in B_0(\Sigma)$ such that $\varphi \geq \psi$. Let $f, g \in \mathcal{F}$ be such that $u(f) = \varphi$ and $u(g) = \psi$. It follows that $u(f(s)) \geq u(g(s))$ for all $s \in S$. By Lemma 8, this implies that $f(s) \succsim^* g(s)$ for all $s \in S$. Since $\succsim^*$ satisfies monotonicity, this implies that $f \succsim^* g$, yielding that $\varphi = u(f) \succeq^* u(g) = \psi$.

4. Consider $k, h \in \mathbb{R}$ and $\varphi \in B_0(\Sigma)$. We first assume that $k > h$ and $k = 0$. By point 3, we have that $\varphi = \varphi + k \succeq^* \varphi + h$. By contradiction, assume that $\varphi \not\succ^* \varphi + h$. It follows that $\varphi \sim^* \varphi + h$, yielding that $I = \{w \in \mathbb{R} : \varphi \sim^* \varphi + w\}$ is a non-empty set which contains 0 and h. We next prove that $I = \mathbb{R}$. First, consider $w_1, w_2 \in I$. Without loss of generality,

assume that $w_1 \geq w_2$. By point 3 and since $w_1, w_2 \in I$, we have that for each $\lambda \in (0,1)$

$$\varphi \succeq^* \varphi + w_1 \succeq^* \varphi + (\lambda w_1 + (1-\lambda) w_2) \succeq^* \varphi + w_2 \succeq^* \varphi$$

proving that $\varphi \sim^* \varphi + (\lambda w_1 + (1-\lambda) w_2)$, that is, $\lambda w_1 + (1-\lambda) w_2 \in I$. Next, we observe that $I \cap (-\infty, 0) \neq \emptyset \neq I \cap (0, \infty)$. Since $h \in I$ and $h < 0$, we have that $I \cap (-\infty, 0) \neq \emptyset$. Since I is an interval and $0, h \in I$, we have that $h/2 \in I$. By point 1 and since $\varphi \sim^* \varphi + h/2$, we have that $\varphi - h/2 \sim^* (\varphi + h/2) - h/2 = \varphi$, proving that $0 < -h/2 \in I \cap (0, \infty)$. By definition of I, note that if $w \in I \backslash \{0\}$, then $\varphi + w \sim^* \varphi$. By point 1 and since $w/2 \in I$ and $\succeq^*$ is a preorder, we have that $(\varphi + w) + w/2 \sim^* \varphi + w/2 \sim^* \varphi$, that is, $\frac{3}{2}w, \frac{1}{2}w \in I$. Since I is an interval, we have that either $\left[\frac{3}{2}w, \frac{1}{2}w\right] \subseteq I$ if $w < 0$ or $\left[\frac{1}{2}w, \frac{3}{2}w\right] \subseteq I$ if $w > 0$. This will help us in proving that I is unbounded from below and above. By contradiction, assume that I is bounded from below and define $m = \inf I$. Since $I \cap (-\infty, 0) \neq \emptyset$, we have that $m < 0$. Consider $\{w_n\}_{n \in \mathbb{N}} \subseteq I \cap (-\infty, 0)$ such that $w_n \downarrow m$. Since $\left[\frac{3}{2}w_n, \frac{1}{2}w_n\right] \subseteq I$ for all $n \in \mathbb{N}$, it follows that $m \leq \frac{3}{2}w_n$ for all $n \in \mathbb{N}$. By passing to the limit, we obtain that $m \leq \frac{3}{2}m < 0$, a contradiction. By contradiction, assume that I is bounded from above and define $M = \sup I$. Since $I \cap (0, \infty) \neq \emptyset$, we have that $M > 0$. Consider $\{w_n\}_{n \in \mathbb{N}} \subseteq I \cap (0, \infty)$ such that $w_n \uparrow M$. Since $\left[\frac{1}{2}w_n, \frac{3}{2}w_n\right] \subseteq I$ for all $n \in \mathbb{N}$, it follows that $M \geq \frac{3}{2}w_n$ for all $n \in \mathbb{N}$. By passing to the limit, we obtain that $M \geq \frac{3}{2}M > 0$, a contradiction. To sum up, I is a non-empty unbounded interval, that is, $I = \mathbb{R}$. This implies that $\varphi \sim^* \varphi + w$ for all $w \in \mathbb{R}$. In particular, select $w_1 = \|\varphi\|_\infty + 1$ and $w_2 = -\|\varphi\|_\infty - 1$. Since $\succeq^*$ is a preorder, we have that $\varphi + w_1 \sim^* \varphi + w_2$. Moreover, $\varphi + w_1 \geq 1 > -1 \geq \varphi + w_2$. By point 3, this implies that $\varphi + w_1 \succeq^* 1 \succeq^* -1 \succeq^* \varphi + w_2$. Since $\succeq^*$ is a preorder and $\varphi + w_1 \sim^* \varphi + w_2$, we can conclude that $1 \sim^* -1$. Note also that there exist $x, y \in X$ such that $u(x) = 1$ and $u(y) = -1$. By Lemma 8, this implies that $x \succ^* y$. By definition of $\succeq^*$ and since $u(x) = 1 \sim^* -1 = u(y)$, we also have that $y \succsim^* x$, a contradiction. Thus, we proved that if $k > h$ and $k = 0$, then $\varphi + k \succ^* \varphi + h$. Assume simply that $k > h$. This implies that $0 > h - k$ and $\varphi \succ^* \varphi + (h - k)$. By point 1, we can conclude that $\varphi + k \succ^* \varphi + (h - k) + k = \varphi + h$.

5. Consider $\varphi, \psi, \xi \in B_0(\Sigma)$ and $\lambda \in (0,1)$. Assume that $\varphi \succeq^* \xi$ and $\psi \succeq^* \xi$. Let $f, g, h \in \mathcal{F}$ be such that $u(f) = \varphi$, $u(g) = \psi$ and $u(h) = \xi$. By assumption and definition of $\succeq^*$, we have that $f \succsim^* h$ and $g \succsim^* h$. Since $\succsim^*$ satisfies convexity and u is affine, this implies that $\lambda f + (1-\lambda) g \succsim^* h$, yielding that $\lambda \varphi + (1-\lambda) \psi = \lambda u(f) + (1-\lambda) u(g) = u(\lambda f + (1-\lambda) g) \succeq^* u(h) = \xi$.

Points 1–5 prove the first part of the statement. Finally, consider $\varphi, \psi \in B_0(\Sigma)$. Note that there exist a partition $\{A_i\}_{i=1}^n \subseteq \Sigma$ of S and $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ in $\mathbb{R}$ such that

$$\varphi = \sum_{i=1}^n \alpha_i 1_{A_i} \text{ and } \psi = \sum_{i=1}^n \beta_i 1_{A_i}$$

Note that $\{s \in S : \varphi(s) \neq \psi(s)\} = \cup_{i \in \{1,\ldots,n\}:\alpha_i \neq \beta_i} A_i$. Since $\varphi \overset{Q}{=} \psi$, we have that $q(A_i) = 0$ for all $q \in Q$ and for all $i \in \{1, ..., n\}$ such that $\alpha_i \neq \beta_i$. Since u is unbounded, define $\{x_i\}_{i=1}^n \subseteq X$ to be such that $u(x_i) = \alpha_i$ for all $i \in \{1, ..., n\}$. Since u is unbounded, define $\{y_i\}_{i=1}^n \subseteq X$ to be such that $y_i = x_i$ for all $i \in \{1, ..., n\}$ such that $\alpha_i = \beta_i$ and $u(y_i) = \beta_i$ otherwise. Define $f, g : S \to X$ by $f(s) = x_i$ and $g(s) = y_i$ for all $s \in A_i$ and for all $i \in \{1, ..., n\}$. It is immediate to see that $f \overset{Q}{=} g$ as well as $u(f) = \varphi$ and $u(g) = \psi$. Since $\succsim^*$ is objectively Q-coherent, we have that $f \sim^* g$, yielding that $\varphi \sim^* \psi$ and proving the second part of the statement. ■

The next three results (Lemmas 10 and 11 as well as Proposition 12) will help us representing $\succeq^*$. This paired with Lemma 8 and Proposition 13 will yield the proof of Lemma 7.

Lemma 10 *Let $\succeq^*$ be a convex niveloidal binary relation. If $\psi \in B_0(\Sigma)$, then $U(\psi) = \{\varphi \in B_0(\Sigma) : \varphi \succeq^* \psi\}$ is a non-empty convex set such that:*

1. *$\psi \in U(\psi)$;*
2. *if $\varphi \in B_0(\Sigma)$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ are such that $k_n \uparrow k$ and $\varphi - k_n \in U(\psi)$ for all $n \in \mathbb{N}$, then $\varphi - k \in U(\psi)$;*
3. *if $k > 0$, then $\psi - k \notin U(\psi)$;*
4. *if $\varphi_1 \geq \varphi_2$ and $\varphi_2 \in U(\psi)$, then $\varphi_1 \in U(\psi)$;*
5. *if $k \geq 0$ and $\varphi_2 \in U(\psi)$, then $\varphi_2 + k \in U(\psi)$.*

Proof Since $\succeq^*$ is reflexive, we have that $\psi \in U(\psi)$, proving that $U(\psi)$ is non-empty and point 1. Consider $\varphi_1, \varphi_2 \in U(\psi)$ and $\lambda \in (0, 1)$. By definition, we have that $\varphi_1 \succeq^* \psi$ and $\varphi_2 \succeq^* \psi$. Since $\succeq^*$ satisfies convexity, we have that $\lambda\varphi_1 + (1 - \lambda)\varphi_2 \succeq^* \psi$, proving convexity of $U(\psi)$. Consider $\varphi \in B_0(\Sigma)$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $k_n \uparrow k$ and $\varphi - k_n \in U(\psi)$ for all $n \in \mathbb{N}$. It follows that $\varphi - k_n \succeq^* \psi$ for all $n \in \mathbb{N}$, then $\varphi - k \succeq^* \psi$, that is, $\varphi - k \in U(\psi)$, proving point 2. If $k > 0$, then $0 > -k$ and $\psi = \psi + 0 \succ^* \psi - k$, that is, $\psi - k \notin U(\psi)$, proving point 3. Consider $\varphi_1 \geq \varphi_2$ such that $\varphi_2 \in U(\psi)$, then $\varphi_1 \succeq^* \varphi_2$ and $\varphi_2 \succeq^* \psi$, yielding that $\varphi_1 \succeq^* \psi$ and, in particular, $\varphi_1 \in U(\psi)$, proving point 4. Finally, to prove point 5, it is enough to set $\varphi_1 = \varphi_2 + k$ in point 4. ■

Before stating the next result, we define few properties that will turn out to be useful later on. A functional $I : B_0(\Sigma) \to \mathbb{R}$ is:

1. a niveloid if $I(\varphi) - I(\psi) \leq \sup_{s \in S}(\varphi(s) - \psi(s))$ for all $\varphi, \psi \in B_0(\Sigma)$;

2. normalized if $I(k) = k$ for all $k \in \mathbb{R}$;

3. monotone if for each $\varphi, \psi \in B_0(\Sigma)$

$$\varphi \geq \psi \implies I(\varphi) \geq I(\psi)$$

4. $\succeq^*$-consistent if for each $\varphi, \psi \in B_0(\Sigma)$

$$\varphi \succeq^* \psi \implies I(\varphi) \geq I(\psi)$$

5. concave if for each $\varphi, \psi \in B_0(\Sigma)$ and $\lambda \in (0,1)$

$$I(\lambda\varphi + (1-\lambda)\psi) \geq \lambda I(\varphi) + (1-\lambda) I(\psi)$$

6. translation invariant if for each $\varphi \in B_0(\Sigma)$ and $k \in \mathbb{R}$

$$I(\varphi + k) = I(\varphi) + k$$

Lemma 11 *Let $\succeq^*$ be a convex niveloidal binary relation. If $\psi \in B_0(\Sigma)$, then the functional $I_\psi : B_0(\Sigma) \to \mathbb{R}$, defined by*

$$I_\psi(\varphi) = \max\{k \in \mathbb{R} : \varphi - k \in U(\psi)\} \qquad \forall \varphi \in B_0(\Sigma)$$

is a concave niveloid which is $\succeq^*$-consistent and such that $I_\psi(\psi) = 0$. Moreover, we have that:

1. *The functional $\bar{I}_\psi = I_\psi - I_\psi(0)$ is a normalized concave niveloid which is $\succeq^*$-consistent.*

2. *If $\succeq^*$ satisfies*

$$\psi \overset{Q}{=} \psi' \implies \psi \sim^* \psi'$$

then

$$\psi \overset{Q}{=} \psi' \implies I_\psi = I_{\psi'} \text{ and } \bar{I}_\psi = \bar{I}_{\psi'}$$

Proof Consider $\varphi \in B_0(\Sigma)$. Define $C_\varphi = \{k \in \mathbb{R} : \varphi - k \in U(\psi)\}$. Note that C_φ is nonempty. Indeed, if we set $k = -\|\varphi\|_\infty - \|\psi\|_\infty$, then we obtain that $\varphi - k = \varphi + \|\varphi\|_\infty + \|\psi\|_\infty \geq 0 + \|\psi\|_\infty \geq \psi \in U(\psi)$. By property 4 of Lemma 10, we can conclude that $\varphi - k \in U(\psi)$, that is, $k \in C_\varphi$. Since $U(\psi)$ is convex, it follows that C_φ is an interval. Since $\varphi \in B_0(\Sigma)$, note that there exists $\hat{k} \in \mathbb{R}$ such that $\psi \geq \varphi - \hat{k}$. It follows that $\psi \succeq^* \varphi - \hat{k}$. In particular, we can conclude that $\psi \succ^* \varphi - \left(\hat{k} + \varepsilon\right)$ for all $\varepsilon > 0$. This yields that C_φ is bounded from above. Finally, assume that $\{k_n\}_{n \in \mathbb{N}} \subseteq C_\varphi$ and $k_n \uparrow k$. By property 2 of Lemma 10, we can conclude that $k \in C_\varphi$. To sum up, C_φ is a non-empty bounded from above interval of $\mathbb{R}$ that satisfies the property

$$\{k_n\}_{n\in\mathbb{N}} \subseteq C_\varphi \text{ and } k_n \uparrow k \implies k \in C_\varphi \tag{61}$$

The first part yields that $\sup\{k \in \mathbb{R} : \varphi - k \in U(\psi)\} = \sup C_\varphi \in \mathbb{R}$ is well defined. By (61), we also have that $\sup C_\varphi \in C_\varphi$, that is, $\sup C_\varphi = \max C_\varphi$, proving that I_ψ is well defined. Next, we prove that I_ψ is a concave niveloid. We first show that I_ψ is monotone and translation invariant. By Proposition 2 of Cerreia-Vioglio et al. (2014), this implies that I_ψ is a niveloid. Rather than proving monotonicity, we prove that I_ψ is $\succsim^*$-consistent.[45] Consider $\varphi_1, \varphi_2 \in B_0(\Sigma)$ such that $\varphi_1 \succsim^* \varphi_2$. By the properties of $\succsim^*$ and definition of I_ψ, we have that

$$\varphi_1 - I_\psi(\varphi_2) \succsim^* \varphi_2 - I_\psi(\varphi_2) \text{ and } \varphi_2 - I_\psi(\varphi_2) \in U(\psi)$$

and, in particular, $\varphi_2 - I_\psi(\varphi_2) \succsim^* \psi$. Since $\succsim^*$ is a preorder, this implies that $\varphi_1 - I_\psi(\varphi_2) \succsim^* \psi$, that is, $\varphi_1 - I_\psi(\varphi_2) \in U(\psi)$ and $I_\psi(\varphi_2) \in C_{\varphi_1}$, proving that $I_\psi(\varphi_1) \geq I_\psi(\varphi_2)$. We next prove translation invariance. Consider $\varphi \in B_0(\Sigma)$ and $k \in \mathbb{R}$. By definition of I_ψ, we can conclude that

$$(\varphi + k) - (I_\psi(\varphi) + k) = \varphi - I_\psi(\varphi) \in U(\psi)$$

This implies that $I_\psi(\varphi) + k \in C_{\varphi+k}$ and, in particular, $I_\psi(\varphi + k) \geq I_\psi(\varphi) + k$. Since k and φ were arbitrarily chosen, we have that

$$I_\psi(\varphi + k) \geq I_\psi(\varphi) + k \qquad \forall \varphi \in B_0(\Sigma), \forall k \in \mathbb{R}$$

This implies that $I_\psi(\varphi + k) = I_\psi(\varphi) + k$ for all $\varphi \in B_0(\Sigma)$ and for all $k \in \mathbb{R}$. We move to prove that I_ψ is concave. Consider $\varphi_1, \varphi_2 \in B_0(\Sigma)$ and $\lambda \in (0,1)$. By definition of I_ψ, we have that

$$\varphi_1 - I_\psi(\varphi_1) \in U(\psi) \text{ and } \varphi_2 - I_\psi(\varphi_2) \in U(\psi)$$

Since $U(\psi)$ is convex, we have that

$$\begin{aligned} &(\lambda\varphi_1 + (1-\lambda)\varphi_2) - (\lambda I_\psi(\varphi_1) + (1-\lambda) I_\psi(\varphi_2)) \\ &= \lambda(\varphi_1 - I_\psi(\varphi_1)) + (1-\lambda)(\varphi_2 - I_\psi(\varphi_2)) \in U(\psi) \end{aligned}$$

yielding that $\lambda I_\psi(\varphi_1) + (1-\lambda) I_\psi(\varphi_2) \in C_{\lambda\varphi_1+(1-\lambda)\varphi_2}$ and, in particular, $I_\psi(\lambda\varphi_1 + (1-\lambda)\varphi_2) \geq \lambda I_\psi(\varphi_1) + (1-\lambda) I_\psi(\varphi_2)$.

Finally, since $\psi \in U(\psi)$, note that $0 \in C_\psi$ and $I_\psi(\psi) \geq 0$. By definition of I_ψ, if $I_\psi(\psi) > 0$, then $\psi - I_\psi(\psi) \in U(\psi)$, a contradiction with property 3 of Lemma 10.

1. It is routine to check that $\bar{I}_\psi$ is a normalized concave niveloid which is $\succsim^*$-consistent.

[45] Since if $\varphi_1 \geq \varphi_2$, then $\varphi_1 \succsim^* \varphi_2$, it follows that $\succsim^*$-consistency implies monotonicity.

2. Clearly, we have that if $\psi \sim^* \psi'$, then $U(\psi) = U(\psi')$, yielding that $I_\psi = I_{\psi'}$ and, in particular, $I_\psi(0) = I_{\psi'}(0)$ as well as $\bar{I}_\psi = \bar{I}_{\psi'}$. The point trivially follows. ■

Proposition 12 *Let $\succeq^*$ be a binary relation on $B_0(\Sigma)$. The following statements are equivalent:*

(i) $\succeq^*$ is convex niveloidal;

(ii) there exists a family of concave niveloids $\{I_\alpha\}_{\alpha\in A}$ on $B_0(\Sigma)$ such that

$$\varphi \succeq^* \psi \iff I_\alpha(\varphi) \geq I_\alpha(\psi) \qquad \forall \alpha \in A \tag{62}$$

(iii) there exists a family of normalized concave niveloids $\{\bar{I}_\alpha\}_{\alpha\in A}$ on $B_0(\Sigma)$ such that

$$\varphi \succeq^* \psi \iff \bar{I}_\alpha(\varphi) \geq \bar{I}_\alpha(\psi) \qquad \forall \alpha \in A \tag{63}$$

Proof (iii) implies (i). It is trivial.

(i) implies (ii). Let $A = B_0(\Sigma)$. We next show that

$$\varphi_1 \succeq^* \varphi_2 \iff I_\psi(\varphi_1) \geq I_\psi(\varphi_2) \qquad \forall \psi \in B_0(\Sigma)$$

where I_ψ is defined as in Lemma 11 for all $\psi \in B_0(\Sigma)$. By Lemma 11, we have that I_ψ is $\succeq^*$-consistent for all $\psi \in B_0(\Sigma)$. This implies that

$$\varphi_1 \succeq^* \varphi_2 \implies I_\psi(\varphi_1) \geq I_\psi(\varphi_2) \qquad \forall \psi \in B_0(\Sigma)$$

Vice versa, consider $\varphi_1, \varphi_2 \in B_0(\Sigma)$. Assume that $I_\psi(\varphi_1) \geq I_\psi(\varphi_2)$ for all $\psi \in B_0(\Sigma)$. Let $\psi = \varphi_2$. By Lemma 11, we have that $I_{\varphi_2}(\varphi_1) \geq I_{\varphi_2}(\varphi_2) = 0$, yielding that $\varphi_1 \geq \varphi_1 - I_{\varphi_2}(\varphi_1) \in U(\varphi_2)$. By point 4 of Lemma 10, this implies that $\varphi_1 \in U(\varphi_2)$, that is, $\varphi_1 \succeq^* \varphi_2$.

(ii) implies (iii). Given a family of concave niveloids $\{I_\alpha\}_{\alpha\in A}$, define $\bar{I}_\alpha = I_\alpha - I_\alpha(0)$ for all $\alpha \in A$. It is immediate to verify that $\bar{I}_\alpha$ is a normalized concave niveloid for all $\alpha \in A$. It is also immediate to observe that

$$I_\alpha(\varphi_1) \geq I_\alpha(\varphi_2) \quad \forall \alpha \in A \iff \bar{I}_\alpha(\varphi_1) \geq \bar{I}_\alpha(\varphi_2) \quad \forall \alpha \in A$$

proving the implication. ■

Remark 1 Given a convex niveloidal binary relation $\succeq^*$ on $B_0(\Sigma)$, we call *canonical* (resp. *canonical normalized*) the representation $\{I_\psi\}_{\psi\in B_0(\Sigma)}$ (resp. $\{\bar{I}_\psi\}_{\psi\in B_0(\Sigma)}$) obtained from Lemma 11 and the proof of Proposition 12. By the previous proof, clearly, $\{I_\psi\}_{\psi\in B_0(\Sigma)}$ and $\{\bar{I}_\psi\}_{\psi\in B_0(\Sigma)}$ satisfy (62) and (63) respectively.

The next result clarifies what the relation is between any representation of $\succeq^*$ and the canonical ones. This will be useful in establishing an extra property of $\left\{\bar{I}_\psi\right\}_{\psi\in B_0(\Sigma)}$ in Corollary 4.

Lemma 12 *Let $\succeq^*$ be a convex niveloidal binary relation. If B is an index set and $\{J_\beta\}_{\beta\in B}$ is a family of normalized concave niveloids such that*

$$\varphi \succeq^* \psi \iff J_\beta(\varphi) \geq J_\beta(\psi) \qquad \forall \beta \in B$$

then for each $\psi \in B_0(\Sigma)$

$$I_\psi(\varphi) = \inf_{\beta\in B}\left(J_\beta(\varphi) - J_\beta(\psi)\right) \qquad \forall \varphi \in B_0(\Sigma) \tag{64}$$

and

$$\bar{I}_\psi(\varphi) = \inf_{\beta\in B}\left(J_\beta(\varphi) - J_\beta(\psi)\right) + \sup_{\beta\in B} J_\beta(\psi) \qquad \forall \varphi \in B_0(\Sigma) \tag{65}$$

Proof Fix $\varphi \in B_0(\Sigma)$ and $\psi \in B_0(\Sigma)$. By definition, we have that

$$I_\psi(\varphi) = \max\{k \in \mathbb{R} : \varphi - k \in U(\psi)\}$$

Since $\{J_\beta\}_{\beta\in B}$ represents $\succeq^*$ and each J_β is translation invariant, note that for each $k \in \mathbb{R}$

$$\begin{aligned}\varphi - k \in U(\psi) &\iff \varphi - k \succeq^* \psi \iff J_\beta(\varphi - k) \geq J_\beta(\psi) \quad \forall \beta \in B \\ &\iff J_\beta(\varphi) - k \geq J_\beta(\psi) \quad \forall \beta \in B \iff J_\beta(\varphi) - J_\beta(\psi) \geq k \quad \forall \beta \in B \\ &\iff \inf_{\beta\in B}\left(J_\beta(\varphi) - J_\beta(\psi)\right) \geq k\end{aligned}$$

By definition of I_ψ and since $\varphi - I_\psi(\varphi) \in U(\psi)$, this implies that $I_\psi(\varphi) = \inf_{\beta\in B}\left(J_\beta(\varphi) - J_\beta(\psi)\right)$. Since φ and ψ were arbitrarily chosen, (64) follows. Since $\bar{I}_\psi = I_\psi - I_\psi(0)$, we only need to compute $-I_\psi(0)$. Since each J_β is normalized, we have that $-I_\psi(0) = -\inf_{\beta\in B}\left(J_\beta(0) - J_\beta(\psi)\right) = -\inf_{\beta\in B}\left(-J_\beta(\psi)\right) = \sup_{\beta\in B} J_\beta(\psi)$, proving (65). ■

Corollary 4 *If $\succeq^*$ is a convex niveloidal binary relation, then $\bar{I}_0 \leq \bar{I}_\psi$ for all $\psi \in B_0(\Sigma)$.*

Proof By Lemma 12 and Remark 1 and since each $\bar{I}_{\psi'}$ is a normalized concave niveloid, we have that

$$\bar{I}_0(\varphi) = \inf_{\psi'\in B_0(\Sigma)}\left(\bar{I}_{\psi'}(\varphi) - \bar{I}_{\psi'}(0)\right) + \sup_{\psi'\in B_0(\Sigma)} \bar{I}_{\psi'}(0) = \inf_{\psi'\in B_0(\Sigma)} \bar{I}_{\psi'}(\varphi) \leq \bar{I}_\psi(\varphi) \quad \forall \varphi \in B_0(\Sigma)$$

for all $\psi \in B_0(\Sigma)$, proving the statement. ■

The next result is instrumental in providing a multi-variational representation of $\succsim^*$ parametrized by Q, when $|Q| \geq 2$. In order to discuss it, we need a piece of terminology. We denote by V the quotient space $B_0(\Sigma)/M$ where M is the vector subspace $\left\{\varphi \in B_0(\Sigma) : \varphi \overset{Q}{=} 0\right\}$. Recall that the elements of V are equivalence classes $[\psi]$ with $\psi \in B_0(\Sigma)$ where $\psi', \psi'' \in [\psi]$ if and only if $\psi \overset{Q}{=} \psi' \overset{Q}{=} \psi''$. Recall that Q is convex.

Proposition 13 *If (S,Σ) is a standard Borel space and $|Q| \geq 2$, then there exists a bijection $f : V \to Q$.*

Proof We begin by observing that:

$$|ca(\Sigma)| \leq |ca_+(\Sigma) \times ca_+(\Sigma)| = |ca_+(\Sigma)| = |(0,\infty) \times \Delta^\sigma| = |\Delta^\sigma|$$

The first inequality holds because the map $g : ca(\Sigma) \to ca_+(\Sigma) \times ca_+(\Sigma)$, defined by $\mu \mapsto (\mu^+, \mu^-)$, is injective. By Theorem 1.4.5 of Srivastava (1998) and since Σ is non-trivial, we have that $ca_+(\Sigma)$ is infinite, yielding that a bijection justifying the first equality exists. As to the second equality, the map $g : ca_+(\Sigma) \setminus \{0\} \to (0,\infty) \times \Delta^\sigma$, defined by $\mu \mapsto (\mu(S), \mu/\mu(S))$, is a bijection and so $|ca_+(\Sigma) \setminus \{0\}| = |(0,\infty) \times \Delta^\sigma|$. By Theorem 1.3.1 of Srivastava (1998), we can conclude that $|ca_+(\Sigma)| = |ca_+(\Sigma) \setminus \{0\}| = |(0,\infty) \times \Delta^\sigma|$. As to the last equality, by Theorem 1.4.5 and Exercise 1.5.1 of Srivastava (1998), being $|(0,\infty)| = |(0,1)| \leq |\Delta^\sigma|$, we have $|\Delta^\sigma| \leq |(0,\infty) \times \Delta^\sigma| = |(0,1) \times \Delta^\sigma| \leq |\Delta^\sigma \times \Delta^\sigma| = |\Delta^\sigma|$, yielding that $|(0,\infty) \times \Delta^\sigma| = |\Delta^\sigma|$.

We conclude that $|ca(\Sigma)| \leq |\Delta^\sigma|$, that is, there exists an injective map $g : ca(\Sigma) \to \Delta^\sigma$. Since Q is a compact and convex subset of Δ^σ, there exists $\bar{q} \in Q$ such that $q \ll \bar{q}$ for all $q \in Q$. We define $h : V \to ca(\Sigma)$ by

$$h([\psi])(A) = \int_A \psi d\bar{q} \qquad \forall A \in \Sigma$$

Note that h is well defined. For, if $\psi' \in [\psi]$, that is, $\psi \overset{Q}{=} \psi'$, then $\psi \overset{\bar{q}}{=} \psi'$, yielding that $\int_A \psi d\bar{q} = \int_A \psi' d\bar{q}$ for all $A \in \Sigma$. Similarly, $h([\psi]) = h([\psi'])$ implies that $\psi \overset{\bar{q}}{=} \psi'$. Since $q \ll \bar{q}$ for all $q \in Q$, this implies that $\psi \overset{Q}{=} \psi'$ and $[\psi] = [\psi']$, proving h is injective. This implies that $\tilde{f} = g \circ h$ is a well defined injective function from V to Δ^σ. Clearly, we have that $|\Delta^\sigma| \geq \left|\tilde{f}(V)\right| \geq |[0,1]|$. Since (S,Σ) is a standard Borel space and Q is convex and $|Q| \geq 2$, we also have that $|[0,1]| \geq |\Delta^\sigma| \geq |Q| \geq |[0,1]|$. This implies that $|V| = \left|\tilde{f}(V)\right| = |Q|$, proving the statement. ■

We can now prove our multi-variational representation result for dominance relations.

Proof of Lemma 7 (ii) implies (i). It is trivial.

(i) implies (ii). Since $\succsim^*$ is a dominance relation, if $|Q| = 1$, that is $Q = \{\bar{q}\}$, then $\succsim^*$ is complete. By Maccheroni et al. (2006) and since $\succsim^*$ is unbounded, it follows that there exists an onto and affine $u : X \to \mathbb{R}$ and a grounded, lower semicontinuous and convex $c_{\bar{q}} : \Delta \to [0,\infty]$

such that $V : \mathcal{F} \to \mathbb{R}$ defined by

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c_{\bar{q}}(p) \right\} \qquad \forall f \in \mathcal{F}$$

represents $\succsim^*$. If we define $c : \Delta \times Q \to [0, \infty]$ by $c(p, q) = c_{\bar{q}}(p)$ for all $(p, q) \in \Delta \times Q$, then we have that c is variational. By Lemma 5 and since $\succsim^*$ is objectively Q-coherent, it follows that $\operatorname{dom} c(\cdot, q) \subseteq \Delta^{\ll}(Q)$ for all $q \in Q$, proving the implication. Assume $|Q| > 1$. By Lemma 8, there exists an onto affine function $u : X \to \mathbb{R}$ which represents $\succsim^*$ on X. By Lemma 9, this implies that we can consider the convex niveloidal binary relation $\succeq^*$ defined as in (60). By definition of $\succeq^*$ and Proposition 12 (and Remark 1), we have that

$$f \succsim^* g \iff u(f) \succeq^* u(g) \iff \bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g)) \quad \forall \psi \in B_0(\Sigma)$$

where each $\bar{I}_\psi$ is a normalized concave niveloid. As before, consider $V = B_0(\Sigma)/M$ where M is the vector subspace $\left\{ \varphi \in B_0(\Sigma) : \varphi \overset{Q}{=} 0 \right\}$. For each equivalence class $[\psi]$, select exactly one $\psi' \in B_0(\Sigma)$ such that $\psi' \in [\psi]$. In particular, let $\psi' = 0$ when $[\psi] = [0]$. We denote this subset of $B_0(\Sigma)$ by $\tilde{V}$. Clearly, we have that

$$\bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g)) \quad \forall \psi \in B_0(\Sigma) \implies \bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g)) \quad \forall \psi \in \tilde{V}$$

Vice versa, assume that $\bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g))$ for all $\psi \in \tilde{V}$. Consider $\hat{\psi} \in B_0(\Sigma)$. It follows that there exists $[\psi]$ in V such that $\hat{\psi} \in [\psi]$. Similarly, consider $\psi' \in \tilde{V}$ such that $\psi' \in [\psi]$. It follows that $\hat{\psi} \overset{Q}{=} \psi'$. By Lemmas 9 and 11 and since $\succsim^*$ is objectively Q-coherent, then $\bar{I}_{\hat{\psi}} = \bar{I}_{\psi'}$, yielding that $\bar{I}_{\hat{\psi}}(u(f)) \geq \bar{I}_{\hat{\psi}}(u(g))$. Since $\hat{\psi}$ was arbitrarily chosen $\bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g))$ for all $\psi \in B_0(\Sigma)$. By construction, observe that there exists a bijection $\tilde{f} : \tilde{V} \to V$. By Proposition 13, we have that there exists a bijection $f : V \to Q$. Define $\bar{f} = f \circ \tilde{f}$. By Corollary 4, if we define $\hat{I}_q = \bar{I}_{\bar{f}^{-1}(q)}$ for all $q \in Q$, then we have that $\hat{I}_{\bar{f}(0)} \leq \hat{I}_q$ for all $q \in Q$ and

$$\begin{aligned} f \succsim^* g &\iff \bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g)) \quad \forall \psi \in B_0(\Sigma) \iff \bar{I}_\psi(u(f)) \geq \bar{I}_\psi(u(g)) \quad \forall \psi \in \tilde{V} \\ &\iff \hat{I}_q(u(f)) \geq \hat{I}_q(u(g)) \quad \forall q \in Q \end{aligned}$$

Since each $\hat{I}_q$ is a normalized concave niveloid, we have that for each $q \in Q$ there exists a function $c_q : \Delta \to [0, \infty]$ which is grounded, lower semicontinuous, convex and such that

$$\hat{I}_q(\varphi) = \min_{p \in \Delta} \left\{ \int \varphi dp + c_q(p) \right\} \qquad \forall \varphi \in B_0(\Sigma)$$

Define $c : \Delta \times Q \to [0, \infty]$ by $c(p, q) = c_q(p)$ for all $(p, q) \in \Delta \times Q$. Clearly, the q-sections of c are grounded, lower semicontinuous and convex and (58) holds. By Lemma 5 and (58) and since $\succsim^*$ is objectively Q-coherent, it follows that $\operatorname{dom} c(\cdot, q) \subseteq \Delta^{\ll}(Q)$ for all $q \in Q$. Finally, recall that

$$c(p,q)=\sup_{\varphi\in B_0(\Sigma)}\left\{\hat{I}_q(\varphi)-\int\varphi dp\right\}\qquad\forall p\in\Delta,\forall q\in Q$$

Since $\hat{I}_{\bar{f}(0)}\leq\hat{I}_q$ for all $q\in Q$, we have that for each $q\in Q$

$$c\left(p,\bar{f}(0)\right)=\sup_{\varphi\in B_0(\Sigma)}\left\{\hat{I}_{\bar{f}(0)}(\varphi)-\int\varphi dp\right\}\leq\sup_{\varphi\in B_0(\Sigma)}\left\{\hat{I}_q(\varphi)-\int\varphi dp\right\}=c(p,q)\quad\forall p\in\Delta$$

Since $c\left(\cdot,\bar{f}(0)\right)$ is grounded, lower semicontinuous and convex and $\bar{f}(0)\in Q$, this implies that $\min_{q\in Q}c(\cdot,q)=c\left(\cdot,\bar{f}(0)\right)$ is well defined and shares the same properties, proving that c is variational. ■

Main criterion with fixed Q We can now state our main representation theorem with Q fixed. To this end, we say that a function $c:\Delta\times Q\to[0,\infty]$ is *uniquely null* if, for all $(p,q)\in\Delta\times Q$, the sets $c_p^{-1}(0)$ and $c_q^{-1}(0)$ are at most singletons. We are now ready to state our first representation result.

Theorem 4 *Let $(S,\Sigma,X,Q,\succsim^*,\succsim)$ be a two-preference decision environment under model uncertainty, where (S,Σ) is a standard Borel space. The following statements are equivalent:*

(i) $\succsim^*$ is an unbounded dominance relation and $\succsim$ is a rational preference that are both Q-coherent and jointly satisfy consistency and caution;

(ii) there exist an onto affine function $u:X\to\mathbb{R}$ and a variational pseudo-statistical distance $c:\Delta\times Q\to[0,\infty]$, with $\operatorname{dom}c_Q\subseteq\Delta^{\ll}(Q)$, such that, for all acts $f,g\in\mathcal{F}$,

$$f\succsim^* g\iff\min_{p\in\Delta}\left\{\int u(f)\,dp+c(p,q)\right\}\geq\min_{p\in\Delta}\left\{\int u(g)\,dp+c(p,q)\right\}\qquad\forall q\in Q\quad(66)$$

and

$$f\succsim g\iff\min_{p\in\Delta}\left\{\int u(f)\,dp+\min_{q\in Q}c(p,q)\right\}\geq\min_{p\in\Delta}\left\{\int u(g)\,dp+\min_{q\in Q}c(p,q)\right\}\qquad(67)$$

If, in addition, c is uniquely null, then it can be chosen to be such that $c(p,q)=0$ if and only if $p=q$.

Proof (i) implies (ii). We proceed by steps. Before starting, we make one observation. By Lemma 7 and since $\succsim^*$ is an unbounded dominance relation which is objectively Q-coherent there exist an onto affine function $u:X\to\mathbb{R}$ and a variational $c:\Delta\times Q\to[0,\infty]$ such that $\operatorname{dom}c(\cdot,q)\subseteq\Delta^{\ll}(Q)$ for all $q\in Q$ (in particular, $\operatorname{dom}c_Q(\cdot)\subseteq\cup_{q\in Q}\operatorname{dom}c(\cdot,q)\subseteq\Delta^{\ll}(Q)$)

and

$$f \succsim^* g \iff \min_{p\in\Delta}\left\{\int u(f)\,dp + c(p,q)\right\} \geq \min_{p\in\Delta}\left\{\int u(g)\,dp + c(p,q)\right\} \quad \forall q \in Q$$

We are left to show that $c_Q : \Delta \to [0,\infty]$ is such that

$$f \succsim g \iff \min_{p\in\Delta}\left\{\int u(f)\,dp + c_Q(p)\right\} \geq \min_{p\in\Delta}\left\{\int u(g)\,dp + c_Q(p)\right\} \tag{68}$$

and $c_Q^{-1}(0) = Q$. To prove this we consider c as in the proof of (i) implies (ii) of Lemma 7. This covers both cases $|Q| = 1$ and $|Q| > 1$. In particular, for each $q \in Q$ define $\hat{I}_q : B_0(\Sigma) \to \mathbb{R}$ by

$$\hat{I}_q(\varphi) = \min_{p\in\Delta}\left\{\int \varphi dp + c(p,q)\right\} \qquad \forall \varphi \in B_0(\Sigma)$$

and recall that there exists $\hat{q}(= \bar{f}(0) \in Q$ when $|Q| > 1)$ such that $c(\cdot,\hat{q}) \leq c(\cdot,q)$, thus $\hat{I}_{\hat{q}} \leq \hat{I}_q$, for all $q \in Q$.

Step 1. $\succsim$ agrees with $\succsim^*$ on X. In particular, $u : X \to \mathbb{R}$ represents $\succsim^*$ and $\succsim$.

Proof of the Step Note that $\succsim^*$ and $\succsim$ restricted to X are continuous weak orders that satisfy risk independence. Moreover, by the observation above, $\succsim^*$ is represented by u. By Herstein and Milnor (1953) and since $\succsim$ is non-trivial, it follows that there exists a non-constant and affine function $v : X \to \mathbb{R}$ that represents $\succsim$ on X. Since $(\succsim^*, \succsim)$ jointly satisfy consistency, it follows that for each $x, y \in X$

$$u(x) \geq u(y) \Longrightarrow v(x) \geq v(y)$$

By Corollary B.3 of Ghirardato et al. (2004), u and v are equal up to an affine and positive transformation, hence the statement. We can set $v = u$. $\square$

Step 2. There exists a normalized and monotone functional $I : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim g \iff I(u(f)) \geq I(u(g))$$

Proof of the Step By the same arguments of Step 5 in the proof of Theorem 1 and since $\succsim$ is a rational preference relation, the statement follows. $\square$

Step 3. $I(\varphi) \leq \inf_{q\in Q} \hat{I}_q(\varphi)$ for all $\varphi \in B_0(\Sigma)$.

Proof of the Step Consider $\varphi \in B_0(\Sigma)$. Since each $\hat{I}_q$ is normalized and monotone and u is onto, we have that $\hat{I}_q(\varphi) \in [\inf_{s\in S}\varphi(s), \sup_{s\in S}\varphi(s)] \subseteq \operatorname{Im} u$ for all $q \in Q$. Since $\varphi \in B_0(\Sigma)$, it follows that there exists $f \in \mathcal{F}$ such that $\varphi = u(f)$ and $x \in X$ such that $u(x) = \inf_{q\in Q}\hat{I}_q(\varphi)$. For each $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $u(x_\varepsilon) = u(x) + \varepsilon$. Since $\inf_{q\in Q}\hat{I}_q(\varphi) = u(x)$, it follows that for each $\varepsilon > 0$ there exists $q \in Q$ such that $\hat{I}_q(u(f)) = \hat{I}_q(\varphi) < u(x_\varepsilon) = \hat{I}_q(u(x_\varepsilon))$, yielding that $f \not\succsim^* x_\varepsilon$. Since $(\succsim^*, \succsim)$ jointly satisfy caution, we have that $x_\varepsilon \succsim f$ for all $\varepsilon > 0$. By Step 2, this implies that

$$u(x) + \varepsilon = u(x_\varepsilon) = I(u(x_\varepsilon)) \geq I(u(f)) = I(\varphi) \quad \forall \varepsilon > 0$$

that is, $\inf_{q \in Q} \hat{I}_q(\varphi) = u(x) \geq I(\varphi)$, proving the step. □

Step 4. $I(\varphi) \geq \inf_{q \in Q} \hat{I}_q(\varphi)$ *for all* $\varphi \in B_0(\Sigma)$.

Proof of the Step Consider $\varphi \in B_0(\Sigma)$. We use the same objects and notation of Step 3. Note that for each $q' \in Q$

$$\hat{I}_{q'}(u(f)) = \hat{I}_{q'}(\varphi) \geq \inf_{q \in Q} \hat{I}_q(\varphi) = u(x) = \hat{I}_{q'}(u(x))$$

that is, $f \succsim^* x$. Since $(\succsim^*, \succsim)$ jointly satisfy consistency, we have that $f \succsim x$. By Step 2, this implies that

$$I(\varphi) = I(u(f)) \geq I(u(x)) = u(x) = \inf_{q \in Q} \hat{I}_q(\varphi)$$

proving the step. □

Step 5. $I(\varphi) = \min_{p \in \Delta} \left\{\int \varphi dp + c_Q(p)\right\}$ *for all* $\varphi \in B_0(\Sigma)$.

Proof of the Step By Steps 3 and 4 and since $\hat{I}_{\hat{q}} \leq \hat{I}_q$ for all $q \in Q$, we have that

$$I(\varphi) = \min_{q \in Q} \hat{I}_q(\varphi) = \hat{I}_{\hat{q}}(\varphi) \qquad \forall \varphi \in B_0(\Sigma)$$

Since $c(\cdot, \hat{q}) = c_Q(\cdot)$, it follows that for each $\varphi \in B_0(\Sigma)$

$$I(\varphi) = \hat{I}_{\hat{q}}(\varphi) = \min_{p \in \Delta} \left\{\int \varphi dp + c(p, \hat{q})\right\} = \min_{p \in \Delta} \left\{\int \varphi dp + c_Q(p)\right\}$$

proving the step. □

Step 6. $c_Q^{-1}(0) = Q$.

Proof of the Step By Steps 2 and 5, we have that $V : \mathcal{F} \to \mathbb{R}$ defined by

$$V(f) = \min_{p \in \Delta} \left\{\int u(f) \, dp + c_Q(p)\right\}$$

represents $\succsim$. By Lemma 4 and since $\succsim$ is subjectively Q-coherent and c_Q is well defined, grounded, lower semicontinuous and convex, we can conclude that $c_Q^{-1}(0) = Q$. □

Thus, (68) follows from Steps 2 and 5 while, by Step 6, $c_Q^{-1}(0) = Q$. This completes the proof.

(ii) implies (i). It is routine.

Next, assume that c is uniquely null. Define the correspondence $\Gamma : Q \rightrightarrows Q$ by

$$\Gamma(q) = \{p \in \Delta : c(p,q) = 0\} = \arg\min c_q$$

Since $c_Q \leq c_q$ for all $q \in Q$ and $c_Q^{-1}(0) = Q$, we have that Γ is well defined. Since c_q is grounded, it follows that $\Gamma(q) \neq \emptyset$ for all $q \in Q$. Since c is uniquely null and c_q is grounded, we have that $c_q^{-1}(0)$ is a singleton, that is,

$$c(p,q) = c(p',q) = 0 \Longrightarrow p = p'$$

This implies that $\Gamma(q)$ is a singleton, therefore Γ is a function. Since $c_Q^{-1}(0) = Q$, observe that

$$\cup_{q\in Q}\Gamma(q) = \cup_{q\in Q} \arg\min c_q = \arg\min c_Q = Q$$

that is, Γ is surjective. Since c is uniquely null, we have that $c_p^{-1}(0)$ is at most a singleton, that is,

$$c(p,q) = c(p,q') = 0 \implies q = q'$$

yielding that Γ is injective. To sum up, Γ is a bijection. Define $\tilde{c} : \Delta \times Q \to [0,\infty]$ by $\tilde{c}(p,q) = c(p,\Gamma^{-1}(q))$ for all $(p,q) \in \Delta\times Q$. Note that $\tilde{c}(\cdot,q)$ is grounded, lower semicontinuous, convex and $\operatorname{dom}\tilde{c}(\cdot,q) \subseteq \Delta^{\ll}(Q)$ for all $q \in Q$ and $\operatorname{dom}\tilde{c}_Q(\cdot) \subseteq \Delta^{\ll}(Q)$. Next, we show that $\tilde{c}_Q = c_Q$. Since c_Q is well defined, for each $p \in \Delta$ there exists $q_p \in Q$ such that

$$\tilde{c}(p,\Gamma(q_p)) = c(p,q_p) = \min_{q\in Q} c(p,q) \leq c(p,q') = \tilde{c}(p,\Gamma(q')) \quad \forall q' \in Q$$

Since Γ is a bijection, we have that $\tilde{c}(p,\Gamma(q_p)) \leq \tilde{c}(p,q)$ for all $q \in Q$. Since p was arbitrarily chosen, it follows that

$$c_Q(p) = \min_{q\in Q} c(p,q) = \tilde{c}(p,\Gamma(q_p)) = \min_{q\in Q}\tilde{c}(p,q) = \tilde{c}_Q(p) \quad \forall p \in \Delta$$

To sum up, $\tilde{c}_Q = c_Q$ and $\tilde{c}_Q^{-1}(0) = c_Q^{-1}(0) = Q$. In turn, since c_Q is grounded, lower semicontinuous and convex, this implies that $\tilde{c}_Q$ is grounded, lower semicontinuous and convex. Since Γ is a bijection, we can conclude that (66) holds with $\tilde{c}$ in place of c and (67) holds with $\tilde{c}_Q$ in place of c_Q.

We are left to show that $\tilde{c}(p,q) = 0$ if and only if $p = q$. Since $c_q^{-1}(0)$ is a singleton for all $q \in Q$ and Γ is a bijection, if $\tilde{c}(p,q) = 0$, then $c(p,\Gamma^{-1}(q)) = 0$, yielding that $p = \Gamma(\Gamma^{-1}(q)) = q$. On the other hand, $\tilde{c}(q,q) = c(q,\Gamma^{-1}(q)) = 0$. We can conclude that $\tilde{c}(p,q) = 0$ if and only if $p = q$, proving the last part of the statement. ■

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