# Exploring Recursive Utility

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## 1 Introduction

We use an approximation method to explore implications of the recursive utility preference specification of Kreps and Porteus (1978) and Epstein and Zin (1989) and counterparts to these preferences that capture concerns about model misspecification. We present formulas for (nonstandard) first and second-order approximations to dynamic, stochastic equilibria for models in which economic agents have such recursive preferences. The approximations build formulations from Schmitt-Grohé and Uribe (2004) and Lombardo and Uhlig (2018), we extend them in way that features the uncertainty contributions more prominently. By design, Implied approximations of stochastic discount factors used to represent market or shadow values reside within the exponential linear quadratic class. This class is known to give tractable formulas for asset valuation over alternative investment horizons. See, for instance, Ang and Piazzesi (2003) and Borovička and Hansen (2014). Moreover, they are applicable to production-based macro-finance models with investment opportunities in alternative forms of capital.

We use the approximations to provide further understanding of the preferences and their implications for asset pricing. This opens the door as well to other connections in the macroeconomics-finance literature in which productive, investment and capital accumulation are central model ingredients. As a central part of our analysis, we capture the important uncertainty preference contribution as a change in the probability distribution of the underlying economic dynamics. We link this change of measure to the robust preferences specifications of Hansen and Sargent (2001) and Anderson et al. (2003). The robust preferences formulations build on a robust control literature initiated by Jacobson (1973) and Whittle (1981).

## 2 Small noise expansion of the state dynamics

We follow Lombardo and Uhlig (2018) by considering the following class of stochastic processes indexed by a scalar perturbation parameter $\mathsf{q}$:[1]

$$X_{t+1}(\mathsf{q}) = \psi\left[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right]. \tag{1}$$

Here $X$ is an $n$-dimensional stochastic process and $\{W_{t+1}\}$ is an i.i.d. normally distributed random vector with conditional mean vector 0 and conditional covariance matrix $I$. We parameterize this family so that $\mathsf{q} = 1$ gives the model of interest.

We denote a zero-order expansion $\mathsf{q} = 0$ limit as:

$$X^0_{t+1} = \psi\left(X^0_t, 0, 0\right), \tag{2}$$

and assume that there exists a second-order expansion of $X_t$ around $\mathsf{q} = 0$:

$$X_t \approx X^0_t + \mathsf{q}X^1_t + \frac{\mathsf{q}^2}{2}X^2_t \tag{3}$$

where $X^1_t$ is a first-order expansion.

In the remainder of this chapter we shall construct instances of the second-order expansion (3) in which the generic random variable $X_t$ is replaced, for example, by the logarithm of consumption, a value function, and so on. In approximation (3), the stochastic processes $X^j$, $j = 0, 1, 2$ are appropriate derivatives of $X$ with respect to the perturbation parameter $\mathsf{q}$.

Processes $X^j_t, j = 0, 1, 2.$ have a recursive structure: the stochastic process $X^0_t$ can be computed first, then the process $X^1_t$ next (it depends on $X^0_t$), and then the process $X^2_t$ (it depends on both $X^0_t$ and $X^1_t$).

We use a prime ($'$) to denote a transpose of a matrix or vector. When we include $x'$ in a partial derivative of a scalar function it means that the partial derivative is a row vector. Consistent with this convention, let $\psi^i_{x'}$, the $i^{th}$ entry of $\psi_{x'}$, denote the row vector of first derivatives with respect to the vector $x$, and similarly for $\psi^i_{w'}$. Since $\mathsf{q}$ is scalar, $\psi^i_{\mathsf{q}}$ is the scalar derivative with respect to $\mathsf{q}$. Derivatives are evaluated at $X^0_t$, which in many examples is invariant over time, unless otherwise stated. This invariance follows when we impose a steady state on the deterministic system.

[1]Lombardo and Uhlig (2018) provide a discussion of how their approach builds on more general perturbation methods as discussed by Holmes (2012) and Judd (1998).

The first-derivative process obeys a recursion

$$
X_{t+1}^1 = \begin{bmatrix} \psi_{x'}^1 \\ \psi_{x'}^2 \\ \vdots \\ \psi_{x'}^n \end{bmatrix} X_t^1 + \begin{bmatrix} \psi_{w'}^1 \\ \psi_{w'}^2 \\ \vdots \\ \psi_{w'}^n \end{bmatrix} W_{t+1} + \begin{bmatrix} \psi_{\mathsf{q}}^1 \\ \psi_{\mathsf{q}}^2 \\ \vdots \\ \psi_{\mathsf{q}}^n \end{bmatrix} \tag{4}
$$

that we can write compactly as the following *a first-order vector autoregression*:

$$
X_{t+1}^1 = \psi_{x'} X_t^1 + \psi_{w'} W_{t+1} + \psi_{\mathsf{q}}
$$

We assume that the matrix $\psi_x'$ is stable in the sense that all of its eigenvalues are strictly less than one in modulus.

It is natural for us to denote second derivative processes with double subscripts. For instance, for the double script used in conjunction with the second derivative matrix of $\psi^i$, the first subscript without a prime ($'$) reports the row location; second subscript with a prime ($'$) reports the column location. Differentiating recursion (4) gives:

$$
\begin{aligned}
X_{t+1}^2 = \; & \psi_{x'} X_t^2 + \begin{bmatrix} X_t^{1'} \psi_{xx'}^1 X_t^1 \\ X_t^{1'} \psi_{xx'}^2 X_t^1 \\ \vdots \\ X_t^{1'} \psi_{xx'}^n X_t^1 \end{bmatrix} + 2 \begin{bmatrix} X_t^{1'} \psi_{xw'}^1 W_{t+1} \\ X_t^{1'} \psi_{xw'}^2 W_{t+1} \\ \vdots \\ X_t^{1'} \psi_{xw'}^n W_{t+1} \end{bmatrix} + \begin{bmatrix} W_{t+1}' \psi_{ww'}^1 W_{t+1} \\ W_{t+1}' \psi_{ww'}^2 W_{t+1} \\ \vdots \\ W_{t+1}' \psi_{ww'}^n W_{t+1} \end{bmatrix} \\
& + 2 \begin{bmatrix} \psi_{\mathsf{q}x'}^1 X_t^1 \\ \psi_{\mathsf{q}x'}^2 X_t^1 \\ \vdots \\ \psi_{\mathsf{q}x'}^n X_t^1 \end{bmatrix} + 2 \begin{bmatrix} \psi_{\mathsf{q}w'}^1 W_{t+1} \\ \psi_{\mathsf{q}w'}^2 W_{t+1} \\ \vdots \\ \psi_{\mathsf{q}w'}^n W_{t+1} \end{bmatrix} + \begin{bmatrix} \psi_{\mathsf{qq}}^1 \\ \psi_{\mathsf{qq}}^2 \\ \vdots \\ \psi_{\mathsf{qq}}^n \end{bmatrix}
\end{aligned} \tag{5}
$$

Recursions (4) and (5) have a linear structure with some notable properties. The law of motion for $X^0$ is deterministic and is time invariant if (1) comes from a stationary $\{X_t\}$ process. The dynamics of $X^2$ are nonlinear only in $X^1$ and $W_{t+1}$; so the stable dynamics for $X^1$ that prevail when $\psi_x$ is a stable matrix imply stable dynamics for $X^2$.

Let $C$ denote consumption and $\widehat{C}$ the logarithm of consumption. Suppose that the logarithm of consumption evolves as:

$$
\widehat{C}_{t+1} - \widehat{C}_t = \kappa(X_t, \mathsf{q} W_{t+1}, \mathsf{q}).
$$

Approximate this process by:

$$\widehat{C}_{t+1} - \widehat{C}_t \approx \widehat{C}^0_{t+1} - \widehat{C}^0_t + \mathsf{q}\left(\widehat{C}^1_{t+1} - \widehat{C}^1_t\right) + \frac{\mathsf{q}^2}{2}\left(\widehat{C}^2_{t+1} - \widehat{C}^2_t\right) \tag{6}$$

where

$$\begin{aligned}
\widehat{C}^0_{t+1} - \widehat{C}^0_t &= \kappa(X^0_t, 0, 0) \equiv \eta^c_0 \\
\widehat{C}^1_{t+1} - \widehat{C}^1_t &= \kappa_{x'} X^1_t + \kappa_{w'} W_{t+1} + \kappa_q \\
\widehat{C}^2_{t+1} - \widehat{C}^2_t &= \kappa_{x'} X^2_t + {X^1_t}' \kappa_{x,x'} X^1_t + 2 {X^1_t}' \kappa_{xw'} W_{t+1} + {W_{t+1}}' \kappa_{ww'} W_{t+1} \\
&\quad + 2\kappa_{q,x'} X^1_t + 2\kappa_{qw'} W_{t+1} + \kappa_{qq}.
\end{aligned}$$

In models with endogenous investment and savings, the consumption dynamics and some of the state dynamics will emerge as the solution to a dynamic stochastic equilibrium model. We use the approximating processes (3) and (6) as inputs into the construction of an approximating continuation value process and its risk-adjusted counterpart for recursive utility preferences.

# 3 Approximating a recursive utility value function

In this section, we construct second-order expansions for components of a continuation value process. This process along with its associated stochastic discount factor process are important constituents of models.

The homogeneous of degree one representation of recursive utility is

$$V_t = \left[(1-\beta)\left(C_t\right)^{1-\rho} + \beta\left(R_t\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} \tag{7}$$

where

$$R_t = \left(\mathbb{E}\left[\left(V_{t+1}\right)^{1-\gamma} \mid \mathfrak{A}_t\right]\right)^{\frac{1}{1-\gamma}}. \tag{8}$$

Notice that in equation (7), $V_t$ is a homogeneous of degree one function of $C_t$ and $R_t$. In equation (8), $R_t$ is a homogeneous of degree one function of another function, namely, $V_{t+1}$ as it varies over date $t+1$ information. In equation (7), $0 < \beta < 1$ is a subjective discount factor and $\rho$ describes attitudes toward intertemporal substitution. Formally, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution. In equation (8), $\gamma$ describes attitudes towards risk.

Continuation values are determined only up to an increasing transformation. For computational and conceptual reasons, we find it advantageous to work with the logarithm $\widehat{V}_t = \log V_t$.

The corresponding recursions for $\widehat{V}_t$ expressed in terms of the logarithm of consumption $\widehat{C}_t$ are

$$\widehat{V}_t = \frac{1}{1-\rho} \log \left[ (1-\beta) \exp \left[ (1-\rho)\widehat{C}_t \right] + \beta \exp \left[ (1-\rho)\widehat{R}_t \right] \right] \tag{9}$$

where

$$\widehat{R}_t = \frac{1}{1-\gamma} \log \mathbb{E} \left( \exp \left[ (1-\gamma)\widehat{V}_{t+1} \right] \mid \mathfrak{A}_t \right). \tag{10}$$

The right side of recursion (9) is the logarithm of a constant elasticity of substitution (CES) function of $\exp(\widehat{C}_t)$ and $\exp(\widehat{R}_t)$.

**Remark 3.1** *The limit of $\widehat{R}_t$ as $\gamma$ approaches 1 is ordinary expected logarithmic utility:*

$$\lim_{\gamma \downarrow 1} \widehat{R}_t = \lim_{\gamma \downarrow 1} \frac{\log E \left( \exp \left[ (1-\gamma)\widehat{V}_{t+1} \right] | \mathfrak{A}_t \right)}{1-\gamma} = \mathbb{E} \left( \widehat{V}_{t+1} | \mathfrak{A}_t \right).$$

Our approach will be to construct small noise expansions for both $\widehat{V}_t$ and $\widehat{R}_t$ and then to assemble them appropriately. Before doing so, we consider a reinterpretation of (10).

## 3.1 Robustness to Model Misspecification

A reinterpretation of the utility recursion and the small-noise expansion approach that we'll deploy comes from recognizing that when $\gamma > 1$, (10) emerges from an instance robust control theory in which $\frac{1}{\gamma-1}$ is a penalty parameter on entropy relative to alternatives that constrains the alternative probability models that a decision maker considers when evaluating consumption processes. This interpretation originated in work by Jacobson (1973) and Whittle (1981) that was extended and reformulated recursively by Hansen and Sargent (1995).

Let the random variable $N_{t+1} \geq 0$ satisfy $\mathbb{E}\left(N_{t+1} \mid \mathfrak{A}_t\right) = 1$ so that it is a likelihood ratio. Think of replacing the expected continuation value $\mathbb{E}\left(\widehat{V}_{t+1} \mid \mathfrak{A}_t\right)$ by the minimized value of the following problem:

$$\min_{N_{t+1} \geq 0, \mathbb{E}(N_{t+1} | \mathfrak{A}_t) = 1} \mathbb{E} \left( N_{t+1} \widehat{V}_{t+1} \mid \mathfrak{A}_t \right) + \xi \mathbb{E} \left( N_{t+1} \log N_{t+1} \mid \mathfrak{A}_t \right) \tag{11}$$

where $\xi$ is a parameter that penalizes departures of $N_{t+1}$ from unity as measured by relative entropy. Conditional relative entropy for an altered conditional probability induced by applying change of measure $N_{t+1}$ is

$$\mathbb{E} \left( N_{t+1} \log N_{t+1} \mid \mathfrak{A}_t \right) \geq 0$$

where, because the function $y \log y$ is convex, the inequality follows from Jensen's inequality. Relative entropy is zero when $N_{t+1} = 1$. The minimizer of problem (11), namely,

$$N^*_{t+1} = \frac{\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{A}_t\right]} \tag{12}$$

"tilts" probabilities towards low continuation values, a version of what Bucklew (2004) calls a stochastic version of Murphy's law. The minimized objective

$$-\xi \log \mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+1}\right) \mid \mathfrak{A}_t\right] = \widehat{R}_t$$

where $\widehat{R}_t$ was given previously by equation (10) if we set $\xi = \frac{1}{\gamma-1}$. The random variable $N^*_{t+1}$ will play a central role in the discussion that follows.

## 3.2 Our expansion protocol

To approximate the recursive utility process we deviate from common practice in macroeconomics by letting the risk aversion or robust parameter in preferences depend on $\mathsf{q}$.

$$\xi = \mathsf{q}\xi_o \qquad \gamma - 1 = \frac{\gamma_o - 1}{\mathsf{q}}$$

The aversion to model misspecification or the aversion to risk moves inversely with the parameter $\mathsf{q}$ when we embed the model of interest within a parameterized family of models. This choice has significant and enlightening consequences for continuation value processes and for the minimizing $N$ process used to alter expectations. It has antecedents in the control theory literature, and it has the virtue that implied uncertainty adjustments occur more prominently at lower-order terms in the approximation.

### 3.2.1 Order-zero

Write the order-zero expansion of (9) as

$$\begin{aligned}
\widehat{V}^0_t &= \frac{1}{1-\rho} \log\left[(1-\beta)\exp\left[(1-\rho)\widehat{C}^0_t\right] + \beta \exp\left[(1-\rho)\widehat{R}^0_t\right]\right] \\
\widehat{R}^0_t &= \widehat{V}^0_{t+1},
\end{aligned}$$

where the second equation follows from noting that randomness vanishes in the limit as q approaches 0.

For order zero, write the consumption growth-rate process as

$$\widehat{C}_{t+1}^{0} - \widehat{C}_{t}^{0} = \eta_c^0.$$

The order-zero approximation of (9) is:

$$\widehat{V}_t^0 - \widehat{C}_t^0 = \frac{1}{1-\rho} \log \left[ (1-\beta) + \beta \exp \left[ (1-\rho) \left( \widehat{V}_{t+1}^0 - \widehat{C}_{t+1}^0 + \eta_c \right) \right] \right]$$

We guess that $\widehat{V}_t^0 - \widehat{C}_t^0 = \eta_{v-c}^0$ and will have verified the guess if the following equation is satisfied

$$\exp \left[ (1-\rho) \left( \eta_{v-c}^0 \right) \right] = (1-\beta) + \beta \exp \left[ (1-\rho) \left( \eta_{v-c}^0 \right) \right] \exp \left[ (1-\rho) \eta_c^0 \right],$$

which implies

$$\exp \left[ (1-\rho) \left( \eta_{v-c}^0 \right) \right] = \frac{1-\beta}{1-\beta \exp \left[ (1-\rho) \eta_c^0 \right]}. \tag{13}$$

Equation (13) determines $\eta_{v-c}^0$ as a function of $\eta_c^0$ and the preference parameters $\rho, \beta$, but not the risk aversion parameter $\gamma$. Specifically,

$$\eta_{v-c}^0 = \frac{\log(1-\beta) - \log\left(1 - \beta \exp\left[(1-\rho)\eta_c^0\right]\right)}{1-\rho} \tag{14}$$

### 3.2.2 Order-one

We temporarily take $\widehat{R}_t^1 - \widehat{C}_t^1$ as given (we'll compute it in section (3.2.3)). We construct a recursion by taking a first-order approximation to the nonlinear utility recursion (9)

$$\widehat{V}_t^1 - \widehat{C}_t^1 = \lambda \left( \widehat{R}_t^1 - \widehat{C}_t^1 \right) \tag{15}$$

where

$$\begin{aligned} \lambda &= \left[ \frac{\beta \exp \left[ (1-\rho) \left( \eta_{v-c} + \eta_c^0 \right) \right]}{(1-\beta) + \beta \exp \left[ (1-\rho) \left( \eta_{v-c} + \eta_c \right) \right]} \right] \\ &= \left[ \frac{\beta \exp \left[ (1-\rho) \eta_c \right]}{(1-\beta) \exp \left[ -(1-\rho) \eta_{v-c} \right] + \beta \exp \left[ (1-\rho) \eta_c \right]} \right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\beta \exp\left[(1-\rho)\eta_c\right]}{1 - \beta \exp\left[(1-\rho)\eta_c\right] + \beta \exp\left[(1-\rho)\eta_c\right]}\right] \\
&= \beta \exp\left[(1-\rho)\eta_c\right]
\end{aligned} \tag{16}
$$

Notice how parameter $\rho$ influences the weight $\lambda$ when $\eta_c \neq 0$, in which case the log consumption process displays growth or decay. When $0 < \rho < 1$, the condition $\lambda < 1$ restricts the parameter $\rho$ relative to the consumption growth rate $\eta_c$ since

$$
(1-\rho)\eta_c < -\log \beta
$$

To facilitate computing some useful limits we construct:

$$
\begin{aligned}
\widetilde{V}_t &= \frac{\widehat{V}_t - \widehat{V}_t^0}{\mathsf{q}} \\
\widetilde{R}_t &= \frac{\widehat{R}_t - \widehat{V}_{t+1}^0}{\mathsf{q}}
\end{aligned} \tag{17}
$$

which we assume remain well defined as $\mathsf{q}$ declines to zero, with limits denoted by $\widetilde{V}_t^0, \widetilde{R}_t^0$. Importantly,

$$
\widetilde{R}_t = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right), \tag{18}
$$

Taking limits as $\mathsf{q}$ declines to zero:

$$
\widehat{R}_t^1 = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right] \mid \mathfrak{A}_t\right)
$$

Subtracting $\widehat{C}_t^1$ from both sides gives:

$$
\widehat{R}_t^1 - \widehat{C}_t^1 = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + (1-\gamma_o)\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right] \mid \mathfrak{A}_t\right) \tag{19}
$$

Substituting formula (19) into the right side of (15) gives the recursion for the first-order continuation value:

$$
\widehat{V}_t^1 - \widehat{C}_t^1 = \left(\frac{\lambda}{1-\gamma_o}\right) \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1\right) + (1-\gamma_o)\left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1\right)\right] \mid \mathfrak{A}_t\right) \tag{20}
$$

**Remark 3.2** *We produce a solution by "guess and verify." Suppose that*

$$\widehat{V}_t^1 - \widehat{C}_t^1 = {\upsilon_1}' X_t^1 + \upsilon_0 \tag{21}$$

*It follows from* (20) *that*

$$\begin{aligned} {\upsilon_1}' =& \lambda \left( {\upsilon_1}' \psi_{x'} + \kappa_{x'} \right) \\ \upsilon_0 =& \lambda \left( \upsilon_0 + {\upsilon_1}' \psi_q + \kappa_q + \frac{(1-\gamma_0)}{2} \left| {\upsilon_1}' \psi_{w'} + \kappa_{w'} \right|^2 \right). \end{aligned} \tag{22}$$

*Deduce the second equation by observing that* $\exp\left[ (1-\gamma_o) \left( \widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1 \right) + (1-\gamma_o) \left( \widehat{C}_{t+1}^1 - \widehat{C}_t^1 \right) \right]$ *is distributed as a log normal. The solutions to equations* (22) *are:*

$$\begin{aligned} \upsilon_1 =& \lambda \left( I - \lambda \psi_{x'} \right)^{-1} \kappa_{x'} \\ \upsilon_0 =& \frac{\lambda}{(1-\lambda)} \left( {\upsilon_1}' \psi_q + \kappa_q \right) + \frac{\lambda (1-\gamma_0)}{2(1-\lambda)} \left| {\upsilon_1}' \psi_{w'} + \kappa_{w'} \right|^2. \end{aligned}$$

*The continuation value has two components. The first is:*

$${\upsilon_1}' X_t^1 + \frac{\lambda}{(1-\lambda)} \left( {\upsilon_1}' \psi_q + \kappa_q \right) = \mathbb{E} \left[ \sum_{j=1}^{\infty} \lambda^j \left( \widehat{C}_{t+j}^1 - \widehat{C}_{t+j-1}^1 \right) \mid \mathfrak{A}_t \right]$$

*and the second is a constant long-run risk adjustment given by:*

$$\frac{\lambda (1-\gamma_o)}{2(1-\lambda)} \left| {\upsilon_1}' \psi_{w'} + \kappa_{w'} \right|^2.$$

*This second term is the the variance of*

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} \lambda^j \left( \widehat{C}_{t+j}^1 - \widehat{C}_{t+j-1}^1 \right) \mid \mathfrak{A}_{t+1} \right] \tag{23}$$

*conditioned on* $\mathfrak{A}_t$ *scaled by* $\frac{\lambda(1-\gamma_o)}{2(1-\lambda)}$.

**Remark 3.3** *The formula for* $\upsilon_1$ *depends on the parameter* $\rho$. *Moreover,* $\upsilon_1$ *has a well defined limit as* $\lambda$ *tends to unity as does the variance of* (23). *This limiting variance:*

$$\lim_{\lambda \to 1} \left| {\upsilon_1}' \psi_{w'} + \kappa_{w'} \right|^2.$$

*converges to the variance of the martingale increment of* $\widehat{C}^1$.

**Remark 3.4** *Consider the logarithm of the risk adjusted continuation value approximated to the first order. Note that from* (21),

$$\widehat{V}^1_{t+1} - \widehat{C}^1_t = \upsilon_1{}' X^1_{t+1} + \upsilon_0 + \kappa_{x'} X^1_t + \kappa_{w'} W_{t+1}.$$

*Substitute this expression into fromula* (19) *and use the formula for the mean of random variable distributed as a log normal to show that*

$$\widehat{V}^1_{t+1} - \widehat{R}^1_t = \left(\upsilon_1{}'\psi_w' + \kappa w'\right) W_{t+1} - \left(\frac{1-\gamma_o}{2}\right) |\upsilon_1{}'\psi_w' + \kappa_{w'}|^2$$

Equation (19) is a standard risk-sensitive recursion applied to log-linear dynamics. For instance, see Tallarini (2000)'s paper on risk-sensitive business cycles and Hansen et al. (2008)'s paper on measurement and inference challenges created by the presence of long-term risk. Both of those papers assumed a logarithmic one-period utility function, so that for them $\rho = 1$.

Here we have instead obtained the recursion as a first-order approximation without necessarily assuming log utility. Allowing for $\rho$ to be different than one shows up in both the order zero and order one approximations as reflected in (14) and (20), respectively. As reflected by formula (20), for the first-order approximation the parameter $\lambda = \beta$ when $\rho = 1$. But otherwise, it is different. Equation (19) also is very similar to a first-order approximation proposed in Restoy and Weil (2011). Like formula (19), Restoy and Weil allow for $\rho \neq 1$. In contrast, our equation has an explicit constant term coming from the risk/robustness adjustment, and we have explicit formula for $\lambda$ that depends on preference parameters and the consumption growth rate.

### 3.2.3 Order two

Differentiating equation (9) a second time gives:

$$\widehat{V}^2_t = (1-\lambda)\widehat{C}^2_t + \lambda \widehat{R}^2_t + (1-\rho)(1-\lambda)\lambda \left(\widehat{R}^1_t - \widehat{C}^1_t\right)^2. \tag{24}$$

Equivalently,

$$\widehat{V}^2_t - \widehat{C}^2_t = \lambda \left(\widehat{R}^2_t - \widehat{C}^2_t\right) + (1-\rho)(1-\lambda)\lambda \left(\widehat{R}^1_t - \widehat{C}^1_t\right)^2.$$

Rewrite transformation (17) as

$$\mathsf{q}\widetilde{V}_t = \widehat{V}_t - \widehat{V}_t^0$$
$$\mathsf{q}\widetilde{R}_t = \widehat{R}_t - \widehat{V}_{t+1}^0$$

Differentiating twice with respect to $\mathsf{q}$ and evaluated at $\mathsf{q} = 0$

$$2\frac{d}{d\mathsf{q}}\widetilde{V}_t + \mathsf{q}\frac{d^2}{d\mathsf{q}^2}\widetilde{V}_t\Big|_{\mathsf{q}=0} = \widetilde{V}_t^1 = \widehat{V}_t^2$$
$$2\frac{d}{d\mathsf{q}}\widetilde{R}_t + \mathsf{q}\frac{d^2}{d\mathsf{q}^2}\widetilde{R}_t\Big|_{\mathsf{q}=0} = \widetilde{R}_t^1 = \widehat{R}_t^2$$

Differentiating (18) with respect to $\mathsf{q}$

$$\frac{d\widetilde{R}_t}{d\mathsf{q}} = \frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]\frac{d\widetilde{V}_{t+1}}{d\mathsf{q}} \mid \mathfrak{A}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right)},$$

and thus

$$\begin{aligned}\widehat{R}_t^2 = 2\widetilde{R}_t^1 &= 2E\left(N_{t+1}^0\widetilde{V}_{t+1}^1 \mid \mathfrak{A}_t\right)\\ &= E\left(N_{t+1}^0\widehat{V}_{t+1}^2 \mid \mathfrak{A}_t\right),\end{aligned} \tag{25}$$

where $N_{t+1}^0$

$$\begin{aligned}N_{t+1}^0 \overset{\text{def}}{=} &\frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}^0\right] \mid \mathfrak{A}_t\right)}\\ = &\frac{\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widehat{V}_{t+1}^1\right] \mid \mathfrak{A}_t\right)}.\end{aligned} \tag{26}$$

Subtracting $\widehat{C}_t^2$ from $\widehat{R}_t^2$ and substituting into (25) gives:

$$\widehat{V}_t^2 - \widehat{C}_t^2 = \lambda\mathbb{E}\left(N_{t+1}^0\left[\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2\right) + \left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2\right)\right] \mid \mathfrak{A}_t\right) + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2. \tag{27}$$

Even if the second-order contribution to the consumption process is zero, there will be nontrivial adjustment to the approximation of $\widehat{V} - \widehat{C}$ because $\left(\widehat{R}^1 - \widehat{C}^1\right)^2$ is different from zero. This term vanishes when $\rho = 1$, and its sign will be different depending on whether $\rho$ is bigger or smaller than one.

**Remark 3.5** *The calculation reported in Remark 3.4 implies that*

$$\log N^0_{t+1} = (1-\gamma_o)\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) = (1-\gamma_o)\left({\upsilon_1}'\psi_{w'} + \kappa_{w'}\right) W_{t+1} - \frac{(1-\gamma_o)^2}{2}|{\upsilon_1}'\psi_w' + \kappa w'|^2$$

*As a consequence, under the change in probability measure induced by $N^0_{t+1}$, $W_{t+1}$ has a mean given by*

$$\mu^0 \stackrel{\text{def}}{=} (1-\gamma_o)\left({\upsilon_1}'\psi_{w'} + \kappa_{w'}\right)'$$

*and with the same covariance matrix given by the identity. This is an approximation to robustness adjustment expressed as an altered distribution of the underlying shocks. It depends on $\gamma_o - 1 = \frac{1}{\xi_o}$ as well as the state dynamics as reflected by $\upsilon_1$ and by the shock exposure vectors $\psi_{w'}$ and $\kappa_{w'}$.*

# 4 Stochastic discount factor process

A stochastic discount factor (SDF) process $S = \{S_t : t \geq 0\}$ tells how a consumer responds to small changes in uncertainty and thereby consequently how a consumer values risky payouts. SDF processes have a variety of uses. First, they provide shadow prices that tell how a consumer's uncertainty aversion shapes marginal valuations of risky assets. Second, they shape first-order conditions for optimally choosing financial and physical investments. Third, they underly tractable formulas for equilibrium asset prices. Fourth, they can help construct Pigouvian taxes for correcting adverse externalities under uncertainty. Fifth, they provide useful tools for assessing effects of small (local) changes in government policies.

To indicate how to deduce an SDF process, we begin by positing that the date zero value of a risky date $t$ consumption payout $\chi_t$ is

$$\pi^t_0(\chi_t) = E\left[\left(\frac{S_t}{S_0}\right)\chi_t \Big| \mathfrak{A}_0\right]. \tag{28}$$

We compute the ratio $\frac{S_t}{S_0}$ that appears in formula (28) by evaluating the slope of an indifference curve that runs through both a baseline consumption process $\{C_t\}_{t=0}^{\infty}$ and a perturbed consumption process

$$(C_0 - P_0(\mathsf{q}), C_1, C_2, \ldots, C_t + \mathsf{q}\chi_t, C_{t+1}, ...)\,.$$

We think of $\mathsf{q}$ as parameterizing an indifference curve, so $P_0(\mathsf{q})$ expresses how much current period consumption must be reduced to keep a consumer on the same indifference curve after we replace $C_t$ by $C_t + \mathsf{q}\chi_t$. We set $\pi_0^t(\chi_t)$ defined in equation (28) equal to the slope of that indifference curve:

$$\pi_0^t(\chi_t) = \frac{d}{d\mathsf{q}} P_0(\mathsf{q})\bigg|_{\mathsf{q}=0}.$$

The one-period increment in the stochastic discount factor process for recursive utility is:

$$\begin{aligned}\frac{S_{t+1}}{S_t} &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \exp\left[(1-\gamma)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \exp\left[(\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \\ &= \beta N^*_{t+1} \exp\left(\widehat{S}_{t+1} - \widehat{S}_t\right) \end{aligned} \tag{29}$$

where

$$\widehat{S}_{t+1} - \widehat{S}_t \stackrel{\text{def}}{=} -\rho\left(\widehat{C}_{t+1} + \widehat{C}_t\right) + (\rho - 1)\left(\widehat{V}_{t+1} - \widehat{C}_{t+1}\right) - (\rho - 1)\left(\widehat{R}_t - \widehat{C}_t\right)$$

where $N^*_{t+1}$ induces the change of probability measure that we described previously as the outcome of robustness problem. (See equation (12).) We will use this second formula in what follows.

**Remark 4.1** *To verify formula* (29)*, we compute a one-period intertemporal marginal rate of substitution. Given the valuation recursions* (9) *and* (10)*, we construct two marginal utilities familiar from CES and exponential utility:*

$$\begin{aligned} mc &= (1-\beta)\,(c)^{-\rho} \exp\left[(\rho-1)\hat{v}\right] \\ m\hat{r} &= \beta \exp\left[(1-\rho)\,(\hat{r} - \hat{v})\,\right] \end{aligned}$$

*From the certainty equivalent formula, we construct the marginal utility of the next-period logarithm of the continuation value:*

$$m\hat{v}^+ = \exp\left[(1-\gamma)\left(\hat{v}^+ - \hat{r}\right)\right]$$

*where the + superscript is used to denote the next-period counterpart. In addition, the next-*

*period marginal utility of consumption is*

$$mc^+ = (1-\beta)\left(c^+\right)^{-\rho} \exp\left[(\rho-1)\hat{v}^+\right]$$

*Putting these four formulas together using the chain rule for differentiation gives a marginal rate of substitution:*

$$\frac{(mr)(mv^+)(mc^+)}{mc} = \beta \left(\frac{c^+}{c}\right)^{-\rho} \exp\left[(1-\gamma)\left(\hat{v}^+ - \hat{r}\right)\right] \exp\left[(\rho-1)\left(\hat{v}^+ - \hat{r}\right)\right].$$

*Now let* $\hat{v}^+ = \widehat{V}_{t+1}$*,* $c^+ = C_{t+1}$*,* $C_t = c$ *and* $\hat{r} = \widehat{R}_t$ *to obtain the formula for the one-period stochastic discount factor* (29)*.*

We approximate $\left[\widehat{S}_{t+1} - \widehat{S}_t\right]$ as

$$\widehat{S}^0_{t+1} - \widehat{S}^0_t \approx \left[\widehat{S}^0_{t+1} - \widehat{S}^0_t\right] + \left[\widehat{S}^1_{t+1} - \widehat{S}^1_t\right] + \frac{1}{2}\left[\widehat{S}^2_{t+1} - \widehat{S}^2_t\right]$$

where

$$\begin{aligned}
\widehat{S}^0_{t+1} - \widehat{S}^0_t &\stackrel{\text{def}}{=} \log\beta - \rho\eta^0_c \\
\widehat{S}^1_{t+1} - \widehat{S}^1_t &\stackrel{\text{def}}{=} -\widehat{C}^1_{t+1} + \widehat{C}^1_t + (\rho-1)\left(\widehat{V}^1_{t+1} - \widehat{C}^1_{t+1}\right) - (\rho-1)\left(\widehat{R}^1_t - \widehat{C}^1_t\right) \\
\widehat{S}^2_{t+1} - \widehat{S}^2_t &\stackrel{\text{def}}{=} -\widehat{C}^2_{t+1} + \widehat{C}^2_t + (\rho-1)\left(\widehat{V}^2_{t+1} - \widehat{C}^2_{t+1}\right) - (\rho-1)\left(\widehat{R}^2_t - \widehat{C}^2_t\right)
\end{aligned}$$

We now consider three different approaches to approximating $N^*_{t+1}$.

## 4.1 Approach 1

Write

$$\begin{aligned}
N^*_{t+1} &= \frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right] \mid \mathfrak{A}_t\right]} \\
&= \frac{\exp\left[(1-\gamma_o)\widetilde{V}_{t+1}\right]}{\exp\left[(1-\gamma_o)\widetilde{R}_t\right]}
\end{aligned}$$

Form the "first-order" approximation:

$$\begin{aligned}\log N^*_{t+1} &\approx (1-\gamma_o)\left[\left(\widetilde{V}^0_{t+1}-\widetilde{R}^0_t\right)+\mathsf{q}\left(\widetilde{V}^1_{t+1}-\widetilde{R}^1_t\right)\right]\\ &= (1-\gamma_o)\left[\left(\widehat{V}^1_{t+1}-\widehat{R}^1_t\right)+\frac{\mathsf{q}}{2}\left(\widehat{V}^2_{t+1}-\widehat{R}^2_t\right)\right]\end{aligned} \tag{30}$$

This approach suggests using the following first-oder approximation for the stochastic discount factor:

$$\begin{aligned}\log S_{t+1}-\log S_t \approx (1-\gamma_o)&\left[\left(\widehat{V}^1_{t+1}-\widehat{R}^1_t\right)+\frac{1}{2}\left(\widehat{V}^2_{t+1}-\widehat{R}^2_t\right)\right]\\ &+\left[\widehat{S}^0_{t+1}-\widehat{S}^0_t\right]+\left[\widehat{S}^1_{t+1}-\widehat{S}^1_t\right]\end{aligned}$$

While the implied $N^*_{t+1}$ approximation is positive, it will not have conditional expectation equal to one. In contrast, the exponential of the first-order contribution $(1-\gamma_o)\left(\widehat{V}^1_{t+1}-\widehat{R}^1_t\right)$ will have conditional expectation equal to one as we have noted previously.

## 4.2 Approach 2

If we were to use a second-order approximation of $N^*_{t+1}$, it would push us outside the class of exponentially quadratic stochastic discount factors. Instead we could combine a first-order approximation of $\log N^*_{t+1}$ with a second-order approximation of $\widehat{S}_{t+1}-\widehat{S}_t$:

$$\begin{aligned}\log S_{t+1}-\log S_t \approx 1-\gamma_o)&\left[\left(\widehat{V}^1_{t+1}-\widehat{R}^1_t\right)+\frac{1}{2}\left(\widehat{V}^2_{t+1}-\widehat{R}^2_t\right)\right]\\ &+\left[\widehat{S}^0_{t+1}-\widehat{S}^0_t\right]+\left[\widehat{S}^1_{t+1}-\widehat{S}^1_t\right]+\left[\widehat{S}^2_{t+1}-\widehat{S}^2_t\right]\end{aligned}$$

which would would preserve the quadratic approximation of $\log S_{t+1}-\log S_t$.

## 4.3 Approach 3

Next consider an alternative modification of Approach 1 given by:

$$\log N^*_{t+1} \approx \frac{\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1}+\widetilde{V}^1_{t+1}\right)\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1}+\widetilde{V}^1_{t+1}\right)\right]\mid \mathfrak{A}_t\right)}$$

$$= \frac{\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right] \mid \mathfrak{A}_t\right)}.$$

By design, exponential counterpart of this approximation will have conditional expectation equal to one. With a little bit of algebraic manipulation, it may be shown that this approximation induces a distributional change for $W_{t+1}$ with a conditional mean that is affine in $X_{t+1}$ and an altered conditional variance that is constant over time.

To understand better this choice of approximation, consider the family of random variables (indexed by $\mathsf{q}$)

$$(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \mathsf{q}\widetilde{V}^1_{t+1}\right) - \log \mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1} + \mathsf{q}\widetilde{V}^1_{t+1}\right)\right] \mid \mathfrak{A}_t\right). \tag{31}$$

The corresponding family of exponentials has conditional expectation one and the $\mathsf{q} = 1$ member is the proposed approximation for $N^*_{t+1}$. Differentiate the family with respect to $\mathsf{q}$:

$$\widetilde{V}^1_{t+1} - \frac{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right]\widetilde{V}^1_{t+1} \mid \mathfrak{A}_t\right)}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\widetilde{V}^0_{t+1}\right] \mid \mathfrak{A}_t\right)} = \widetilde{V}^1_{t+1} - \widetilde{R}^1_t.$$

Thus this family of random variables has the same first-order approximation in $\mathsf{q}$ as $\log N^*_{t+1}$ given in (30) and it remains within the linear-quadratic in logarithms formulation.

As a first-order change of probability measure, this approximation will induce state dependence in the conditional mean and will alter the covariance matrix of the shock vector. We find this approach interesting because it links back directly to the outcome of the robustness formulation we described in Section 3.1. Moreover, the state dependence in the mean will induce a corresponding state dependence in the one-period uncertainty prices.

# 5 Long-run risk example

We consider a model with long-run risk components to consumption as suggested by Bansal and Yaron (2004). For the moment, we abstract from production; but as we will see later there is a production counterpart in consumption displays long-run risk. For now think simply specify a consumption process with a long-run risk component.

## 5.1 Approximation

By applying this approximation to the Bansal and Yaron (2004) model, we obtain the state dynamics:

$$
\begin{aligned}
X^0_{t+1} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
X^1_{t+1} &= \begin{bmatrix} \theta^x_{11} & 0 \\ 0 & \theta^x_{22} \end{bmatrix} X^1_t + \begin{bmatrix} \sigma^x_{11} & 0 & 0 \\ 0 & \sigma^x_{22} & 0 \end{bmatrix} W_{t+1} \\
X^2_{t+1} &= \begin{bmatrix} \theta^x_{11} & 0 \\ 0 & \theta^x_{22} \end{bmatrix} X^2_t + \begin{bmatrix} 2X^1_{2,t}\sigma^x_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W_{t+1},
\end{aligned}
$$

where $0 < \theta^x_{11} < 1$ and $0 < \theta^x_{22} < 1$, and the consumption dynamics:

$$
\begin{aligned}
\widehat{C}^0_{t+1} - \widehat{C}^0_t &= \eta^0_c \\
\widehat{C}^1_{t+1} - \widehat{C}^1_t &= \theta^c_1 X^1_{1,t} + \begin{bmatrix} 0 & 0 & \sigma^c_3 \end{bmatrix} W_{t+1} \\
\widehat{C}^2_{t+1} - \widehat{C}^2_t &= \theta^c_1 X^2_{1,t} + \begin{bmatrix} 0 & 0 & 2X^1_{2,t}\sigma^c_3 \end{bmatrix} W_{t+1}.
\end{aligned}
$$

Thus we take consumption to evolve (apprioximately) as:

$$
\begin{aligned}
\widehat{C}_{t+1} - \widehat{C}_t &= \left( \widehat{C}^0_{t+1} + \widehat{C}^1_{t+1} + \frac{1}{2}\widehat{C}^2_{t+1} \right) - \left( \widehat{C}^0_t + \widehat{C}^1_t + \frac{1}{2}\widehat{C}^2_t \right) \\
&= \eta^0_c + \theta^c_1 \left( X^1_{1,t} + \frac{1}{2} X^2_{1,t} \right) + \begin{bmatrix} 0 & 0 & \left(1 + X^1_{2,t}\right)\sigma^c_3 \end{bmatrix} W_{t+1}.
\end{aligned}
$$

The processes $\{X^1_{1,t}\}$ and $\{X^2_{1,t}\}$ contribute temporal dependence to the consumption growth dynamics. The process $\{X^1_{2,t}\}$ contributes stochastic volatility to the consumption dynamics while the stationary specification of the process $\{X^2_{2,t}\}$ is identically zero and can be ignored.

## 5.2 VAR approach

The Markov process governing the predictable component of macroeconomic growth is scalar in the Bansal and Yaron (2004) analysis. Motivated by empirical evidence, Hansen et al. (2008) study an extension of this model where $X^1$ is a vector autoregression. To relate to the VAR approach of Hansen et al., write the first-order approximation for the logarithm of consumption as:

$$\widehat{C}_{t+1}^1 - \widehat{C}_t = \eta_c^0 + \mathbb{D}X_t^1 + \mathbb{F}'W_{t+1}$$

where

$$\kappa_{x'} = \mathbb{D}$$
$$\kappa_{w'} = \mathbb{F}.$$

Where the first-order process $X^1$ includes a predictable component of the macroeconomic growth-rate process and evolves as an autoregression:

$$X_{t+1}^1 = \mathbb{A}X_t^1 + \mathbb{B}W_{t+1},$$

where

$$\psi_{x'} = \mathbb{A}$$
$$\kappa_{w'} = \mathbb{B}.$$

and $\mathbb{A}$ is a stable matrix. Thus the first-order approximation to the Bansal and Yaron (2004) for the consumption dynamics is a special case of the formulation in Hansen et al. (2008).The Hansen et al. (2008) predictability evidence turned out to be "fragile" and was modified and updated in Hansen and Sargent (2021) Appendix B. This same appendix suggests a way to deduce a statistical approximation to the first order dynamics of Bansal and Yaron (2004) from a more general VAR representation of the consumption dynamics.

The row vector $\mathbb{F}$ and matrix $\mathbb{B}$ are configured so that the components of the shock vector $W_{t+1}$ directly disturbs growth in the logarithm of consumption and its predictable(first-order) growth component $X^1$. Notice, in particular that the conditional mean of $\widehat{C}_{t+j} - \widehat{C}_t$ is

$$j\eta_c^0 + \mathbb{D}\left(X_t + \mathbb{A}X_t + ... + \mathbb{A}^{j-1}\right)X_t.$$

The corresponding multi-period forecast errors contribute to the variance of $\widehat{C}_{t+j} - \widehat{C}_t$ with a variance that increases with the horizon. When the process $\{\mathbb{D}X_t\}$ is highly persistent, there is said to be substantial "long-run risk" in consumption.

### 5.3 Approximating continuation values

Returning the original Bansal and Yaron (2004) specification, we consider the approximation of continuation values and the corresponding change in probabilities. The first-order continuation-value approximation is

$$\begin{aligned}\widehat{V}_t^1 - \widehat{C}_t^1 &= \left(\frac{\lambda}{1-\lambda\theta_{11}^x}\right)\theta_1^c X_{1,t}^1 \\ \widehat{R}_t^1 - \widehat{C}_t^1 &= \left(\frac{1}{1-\lambda\theta_{11}^x}\right)\theta_1^c X_{1,t}^1\end{aligned}$$

The implied change in probability measure is

$$N_{t+1}^0 = \exp\left(\mu^0 \cdot W_{t+1} - \frac{1}{2}\left|\mu^0\right|^2\right)$$

where

$$\mu^0 = (1-\gamma_o)\begin{bmatrix}\left(\frac{\lambda}{1-\lambda\theta_{11}^x}\right)\theta_1^c\sigma_{11}^x \\ 0 \\ \sigma_3^c\end{bmatrix}$$

is the implied mean distortion. The negative of $\mu_0$ gives the vector of one period shock exposure prices.

We use formula, (27), for the second-order adjustment. As a first step we compute

$$\mathbb{E}\left[N_{t+1}^0\left(\widehat{C}_{t+1}^2 - \widehat{C}_t^2\right) \mid \mathfrak{A}_t\right] = \theta_c X_{1,t}^2 + 2(\sigma_3)^2 X_{2,t}^1.$$

Thus we are lead to solve:

$$\begin{aligned}\widehat{V}_t^2 - \widehat{C}_t^2 =& \lambda\mathbb{E}\left[N_{t+1}^1\left(\widehat{V}_{t+1}^2 - \widehat{C}_{t+1}^2\right) \mid \mathfrak{A}_t\right] \\ &+ \theta_c X_{1,t}^2 + 2(\sigma_3)^2 X_{2,t}^1 + (1-\rho)(1-\lambda)\lambda\left[\left(\frac{1}{1-\lambda\theta_{11}^x}\right)\theta_1^c X_{1,t}^1\right]^2\end{aligned}$$

forward using the change of probability measure under which the conditional expectation of $W_{t+1}$ is equal to $\mu^0$.

## 5.4 Shock elasticities

We use the shock elasticities to explore pricing implications of this recursive utility specification. We conduct this exploration using the original parameter calibration in Bansal and Yaron (2004). These elasticities and their relation to impulse-response functions introduced first to macroeconomics by Frisch (1933) is described Borovička et al. (2014). In what follows, we use exponential/linear/quadratic implementation by Borovička and Hansen (2014) and by Borovička and Hansen (2016).

Figure 1 gives the shock exposure elasticities for consumption to each of the three shocks. This can can interpreted as nonlinear local impulse responses for consumption (in levels not logarithms). The elasticities for the growth rate shock and the stochastic volatility shock start small and increase over the time horizon as dictated by the persistence of the two exogenous state variable processes. The elasticities for the direct shock to consumption are flat over the horizon as to be expected since the shock directly impacts log consumption in a manor that is permanent. Notice that while elasticities for the volatility shock are different from zero, their contribution is much smaller than the other shocks.[2] Nevertheless, for this Bansal and Yaron (2004) calibration of the long run risk model, stochastic volatility induces state dependence in the elasticities for growth rate and consumption shocks as reflected by quantiles given in the figures.

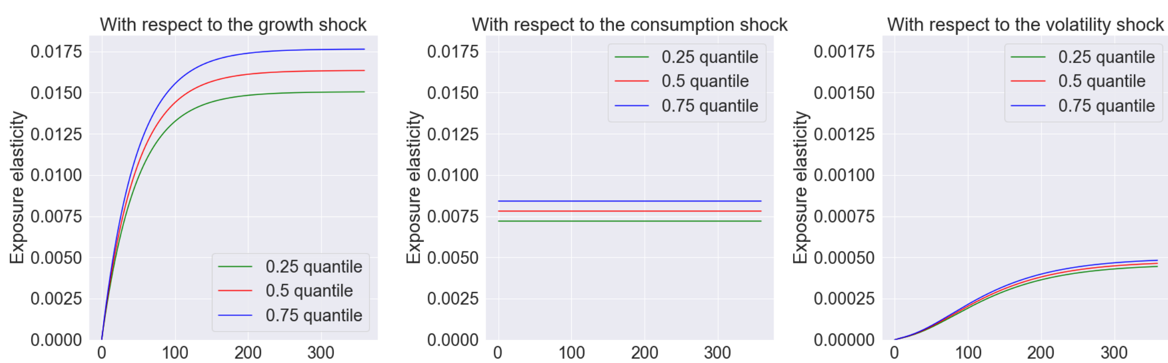


Figure 1: Exposure elasticities for three shocks. The time scale is in months.

Figure 2 gives the corresponding shock price elasticities for $\rho = 2/3$ and $\gamma = 10$. The recursive utility preferences are forward looking as reflected by the continuation-value contribution to the one-period increment to the stochastic discount factor process as given in

[2] It is notable that we are looking at levels and not logarithms of consumption. the local impulse response for the logarithms of consumption is in fact zero for the stochastic volatility shock.

(29). This forward-looking contribution is reflected in shock price elasticities that are now flat for both the growth rate shock and the shock to stochastic volatility. The magnitudes are substantially higher for the shock-price elasticities. While the relative magnitudes are very different, the shock price elasticities are much smaller than the other elasticities.[3].

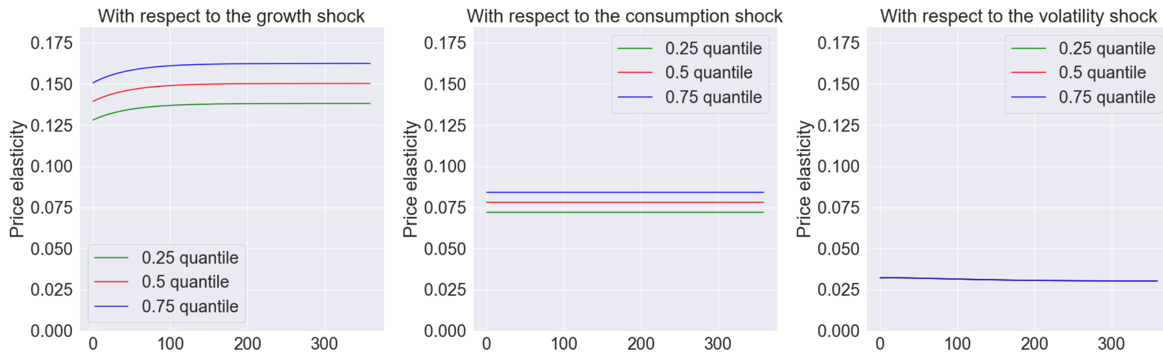


Figure 2: Price elasticities for three shocks. $\rho = 2/3, \gamma =, 10$ $\beta = .998$. The time scale is in months.

Figures 3 and 4 provide the analogous plots for $\rho = 1, 1.5$. The shock price elasticities are very similar given these modes increases in $\rho$. It is evidently the risk aversion parameter $\gamma = 10$ that is important for determining the magnitude of these elasticities. Figure 5 sets $\rho = \gamma = 10$ which corresponds to preferences that are time separable. The forward-looking component to the stochastic discount factor is shut down as is evident from formula (29). Now the shock price elasticities and shock exposure elasticities show a very similar trajectory except that the shock price elasticities are about ten times larger. The stochastic volatility shock price is increased by about seventy-five times. Notice that for longer horizons the $\gamma = \rho = 10$ preference model has prices that are very similar in magnitude to the recursive preference models with more modest specifications of $\rho$.

[3]We normalized the stochastic volatility shock $\sigma_x^2$ to be negative implying that a positive shock reduces the stochastic volatility state variable. Under this normalization, the shock price elasticities are positive.

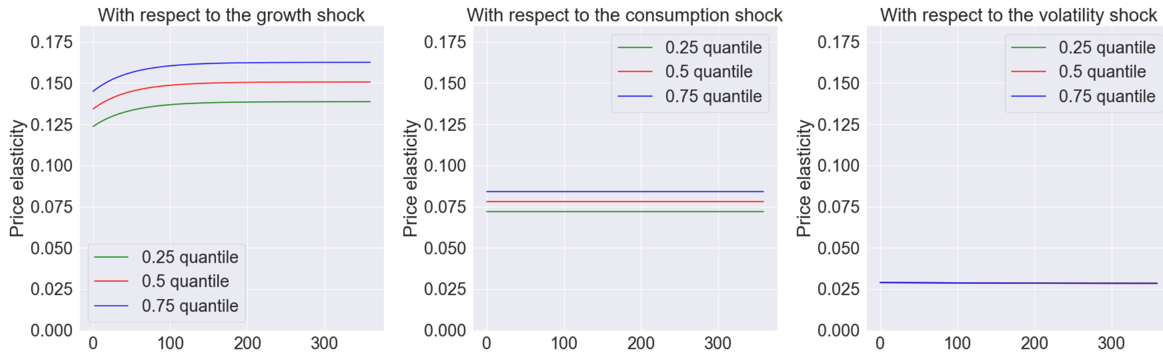


Figure 3: Price elasticities for three shocks. $\rho = 1, \gamma = 10$, $\beta = .998$. The time scale is in months.

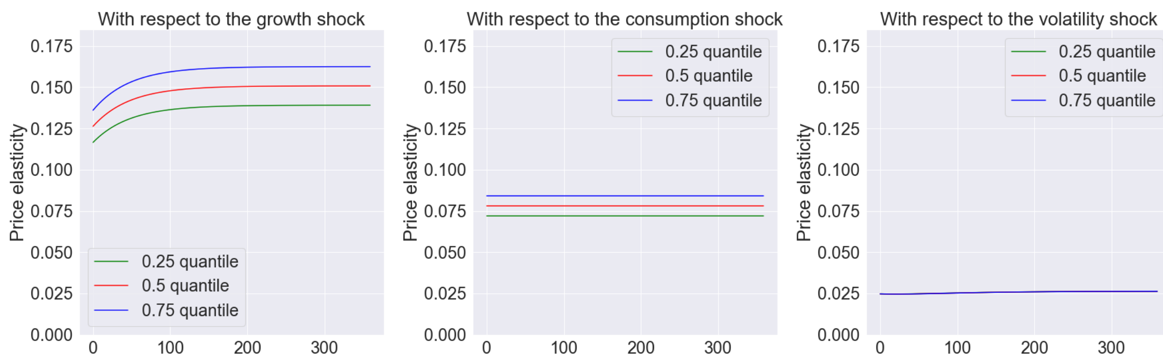


Figure 4: Price elasticities for three shocks. $\rho = 3/2, \gamma = 10$, $\beta = .998$. The time scale is in months.

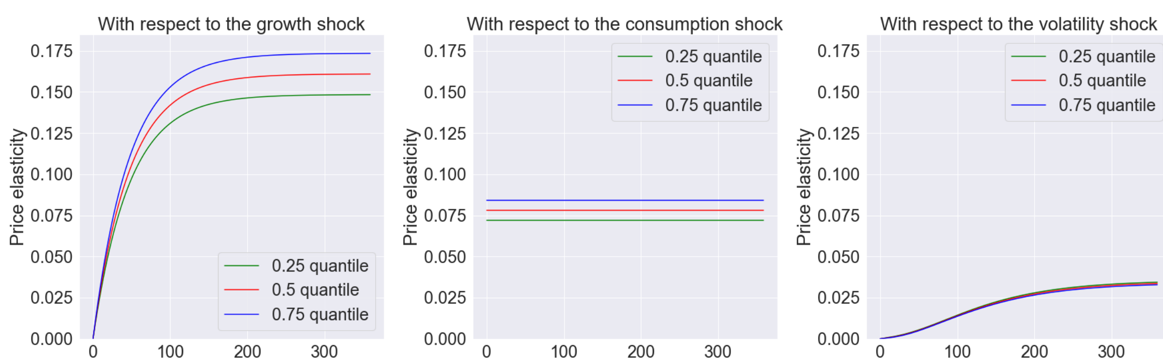


Figure 5: Price elasticities for three shocks. $\rho = 10, \gamma = 10$, $\beta = .998$. The time scale is in months.

# 6 Solving models

The Bansal and Yaron (2004) example along with may others building connections between the macro economy and asset value take aggregate consumption as pre-specified. As we open the door to a richer collection of macroeconomic models, it becomes important to entertain more endogeneity, including investment and other variables familiar to macroeconomics. In this section, we briefly describe one way to extend the approach that builds directly on previous second-order approaches of Kim et al. (2008), Schmitt-Grohé and Uribe (2004), and Lombardo and Uhlig (2018). While such methods should not be viewed as being generically applicable to nonlinear stochastic equilibrium models, we find them useful pedagogically and often as at least initial steps to understanding models that are arguably "smooth." See Pohl et al. (2018) for a careful study of nonlinearity in asset pricing models with recursive utility.[4]

We implement these methods for second-order approximation using the following steps.

i) Given an approximation for $N^*_{t+1}$ and the first and second-order approximations for $\widehat{V}_{t+1} - \widehat{R}_t$, compute first and second-order expansions following previous literature on second-order approximation and solve for the jump variables using change of probability measure implied by the pre-specified $N^*_{t+1}$. We use the approximation for $(\rho-1)\widehat{V}_{t+1} - \widehat{R}_t$ as a plug in for the construction of the logarithm of the stochastic discount factor. Specifically, we use the first-order plug in approximation for $\widehat{V}_{t+1} - \widehat{R}_t$ in the first-order approximation of the stochastic equilibrium and the second-order plug in approximation for $\widehat{V}_{t+1} - \widehat{R}_t$ for the second-order approximation of the stochastic equilibrium.

ii) Compute first and second-order expansions for approximation of $\widehat{V} - \widehat{C}$ and $\widehat{R} - \widehat{C}$ form the approximated consumption dynamics of step ii), and we form the first and second-order approximations for $\widehat{V}_{t+1} - \widehat{R}_t$.

iii) Compute

$$N_{t+1} = \frac{\exp\left[(1-\gamma_o)\left[\widehat{V}^1_{t+1} - \widehat{R}^1_t + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\widehat{V}^1_{t+1} - \widehat{R}^1_t + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right] \mid \mathfrak{A}_t\right)},$$

iv) Set $N^*_{t+1} = N_{t+1}$.

[4]Pohl et al. provide examples of when log-linear or local methods of computation fail to provide good approximations.

The counterpart for a first-order approximation for the stochastic equilibrium uses first-order approximations only for $N^*_{t+1}$ and $\widehat{V}_{t+1} - \widehat{R}_t$ and converges in two steps. In contrast to the usual first-oder approximation, the change of measure will add constant terms to the first-order contribution. The can be viewed as a "precautionary" contribution from recursive utility motivated based on either risk or robustness considerations.

For another second-order approach for solving a stochastic equilibrium, see Appendix A.

# A Some additional calculations

For the purposes of this appendix, write:

$$\frac{S_{t+1}}{S_t} = N^*_{t+1} Q_{t+1} \beta \exp\left[-\rho \left(\log C_{t+1} - \log C_t\right)\right]$$

where

$$\begin{aligned} N^*_{t+1} &= \exp\left[(1-\gamma_o)\left(\widetilde{V}_{t+1} - \widetilde{R}_t\right)\right] \\ Q_{t+1} &= \exp\left[(\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right] \end{aligned}$$

are terms that are contributed by recursive utility.

## A.1 $N^*_{t+1}$ derivatives

$$\begin{aligned} N^0_{t+1} &\overset{\text{def}}{=} \exp\left[(1-\gamma_o)\left(\widetilde{V}^0_{t+1} - \widetilde{R}^0_t\right)\right] = \exp\left[(1-\gamma_o)\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right)\right] \\ N^1_{t+1} &\overset{\text{def}}{=} \frac{d}{d\mathsf{q}} \exp\left[(1-\gamma_o)\left(\widetilde{V}_{t+1} - \widetilde{R}_t\right)\right]\Bigg|_{\mathsf{q}=0} \\ &= N^0_{t+1}(1-\gamma_o)\left(\widetilde{V}^1_{t+1} - \widetilde{R}^1_t\right) = N^0_{t+1}\left(\frac{1-\gamma_o}{2}\right)\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right). \end{aligned}$$

It may be directly verified that $N^1_{t+1}$ has conditional expectation equal to zero.

## A.2 $Q_{t+1}$ derivatives

$$\begin{aligned} Q^0_{t+1} &\overset{\text{def}}{=} \exp\left[(\rho-1)\left(\widehat{V}^0_{t+1} - \widehat{R}^0_t\right)\right] = 1 \\ Q^1_{t+1} &\overset{\text{def}}{=} \frac{d}{d\mathsf{q}} \exp\left[(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right]\Bigg|_{\mathsf{q}=0} = (\rho-1)\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) \\ Q^2_{t+1} &= \frac{d^2}{d\mathsf{q}^2} \exp\left[(\rho-1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)\right]\Bigg|_{\mathsf{q}=0} = (\rho-1)^2\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right)^2 + (\rho-1)\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right) \end{aligned}$$

## A.3 Approximating expectation equations

Consider the equation:

$$\mathbb{E}\left(N_{t+1} Q_{t+1} H_{t+1} \mid \mathfrak{F}_t\right) = 0.$$

where $\beta \exp\left[-\rho\left(\log C_{t+1} - \log C_t\right)\right]$ is absorbed into the construction of $H_{t+1}$.

The the order zero approximation of the product: $N_{t+1}Q_{t+1}H_{t+1}$ is:

$$N^0_{t+1}H^0_{t+1}$$

where we have substituted $Q^0_{t+1} = 1$. Thus the order zero approximate equation is:

$$\mathbb{E}\left(N^0_{t+1}H^0_{t+1} \mid \mathfrak{F}_t\right) = H^0_{t+1} = 0$$

since $N^0_{t+1}$ has conditional expectation equal to one.

The order one approximation of the product: $N_{t+1}Q_{t+1}H_{t+1}$ is:

$$N^1_{t+1}H^0_{t+1} + N^0_{t+1}Q^1_{t+1}H^0_{t+1} + N^0_{t+1}H^1_{t+1}.$$

where we have substituted $Q^0_{t+1} = 1$. Thus the order one approximate equation is:

$$\mathbb{E}\left(N^1_{t+1}H^0_{t+1} + N^0_{t+1}Q^1_{t+1}H^0_{t+1} + N^0_{t+1}H^1_{t+1} \mid \mathcal{F}_t\right) = \mathbb{E}\left(N^0_{t+1}H^1_{t+1} \mid \mathfrak{F}_t\right) = 0$$

where we used the implication that $\mathbb{E}(N^1_{t+1} \mid \mathfrak{F}_t) = 0$ and $H_{t+1} = 0$.

The order two approximation of the product: $N_{t+1}Q_{t+1}H_{t+1}$ is:

$$2N^1_{t+1}H^0_{t+1} + 2N^0_{t+1}Q^1_{t+1}H^1_{t+1} + 2N^1_{t+1}H^1_{t+1} + N^2_{t+1}H^0_{t+1} + N^0_{t+1}Q^2_{t+1}H^0_{t+1} + N^0_{t+1}H^2_{t+1}.$$

Taking expectations and using the same simplifications that we used for the order one approximation, the order two approximate equation is

$$2\mathbb{E}\left(N^0_{t+1}Q^1_{t+1}H^1_{t+1} + N^1_{t+1}H^1_{t+1} \mid \mathfrak{F}_t\right) + \mathbb{E}\left(N^0_{t+1}H^2_{t+1} \mid \mathfrak{F}_t\right) = 0.$$

where we have use that

$$\begin{aligned} H^0_{t+1} &= 0 \\ \mathbb{E}\left(N^2_{t+1}H^0_{t+1} \mid \mathfrak{F}_t\right) &= 0. \end{aligned}$$

The term $\mathbb{E}\left(N^0_{t+1}H^2_{t+1} \mid \mathfrak{F}_t\right)$ coincides with the second order approximation term abstracting from recursive utility but evaluated under the change of measure induced by $N^0_{t+1}$. For our second order adjustment, we need to include the additional term:

$$2\mathbb{E}\left(N^0_{t+1}Q^1_{t+1}H^1_{t+1} + N^1_{t+1}H^1_{t+1} \mid \mathfrak{F}_t\right). \tag{32}$$

To provide usable calculations for (32), write

$$2N^1_{t+1}H^1_{t+1} = (1-\gamma_o)N^0_{t+1}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)H^1_{t+1},$$

Note that under the $N^0_{t+1}$ change of probability measure $\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)$ has conditional mean zero. Express

$$\begin{aligned}
\widehat{V}^1_{t+1} - \widehat{R}^1_t &= \frac{1}{1-\gamma_o}\left(\mu^0\cdot\left(W_{t+1}-\mu^0\right) + \frac{1}{2}\mu^0\cdot\mu^0\right) \\
\widehat{V}^2_{t+1} - \widehat{R}^2_t &= \frac{1}{2}\left(W_{t+1}-\mu^0\right)'\Lambda^2_2\left(W_{t+1}-\mu^0\right) + \left(W_{t+1}-\mu^0\right)'\left(\Lambda^2_1X^1_t + \Lambda^2_0\right),
\end{aligned} \tag{33}$$

and express $H^1_{t+1}$ as

$$H^1_{t+1} = \Gamma_0 + \Gamma_1X^1_t + \Gamma_2\left(W_{t+1}-\mu^0\right).$$

Then

$$2\mathbb{E}\left(N^0_{t+1}Q^1_{t+1}H^1_{t+1} \mid \mathfrak{F}_t\right) = 2\left(\frac{\rho-1}{1-\gamma_o}\right)\left[\Gamma_2\mu^0 + \frac{1}{2}\mu^0\cdot\mu^0\left(\Gamma_0+\Gamma_1X^1_t\right)\right], \tag{34}$$

and

$$2\mathbb{E}\left(N^1_{t+1}H^1_{t+1} \mid \mathfrak{F}_t\right) = (1-\gamma_o)\left(\Lambda_1X^1_t + \Lambda_0\right)'\Gamma_2. \tag{35}$$

Thus expression (32) is affine in $X^1_t$.

We implement these methods for second-order approximation using the following steps.

i) Solve $H^0_{t+1}$ for order zero state and jump variables.

ii) Given a $\mu^0$ compute the first-order expansion following previous literature based on

$$\mathbb{E}\left(N^0_{t+1}H^1_{t+1} \mid \mathfrak{A}_t\right) = 0.$$

iii) Given first and second-order approximations for $\widehat{V}_{t+1} - \widehat{R}_t$, use formulas (34) and (35) to compute (32) as function of the first-order state vector approximation and its evolution as an outcome of the first order model solution from step ii).

iv) Combine the term from iii) with

$$\mathbb{E}\left(N_{t+1}^{0} H_{t+1}^{2} \mid \mathfrak{F}_{t}\right)$$

to compute the second order contribution to the state and jump vectors by following standard methods.

v) Compute $\mu_0$ along with first and second order approximations for $\widehat{V}_{t+1} - \widehat{R}_t$. Use these and return to step ii) until convergence.

# References

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent. 2003. A quartet of semi-groups for model specification, robustness,prices of risk, and model detection. *Journal of the European Economic Association* 1:68–123.

Ang, Andrew and Monika Piazzesi. 2003. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics* 50:745–787.

Bansal, Ravi and Amir Yaron. 2004. Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance* 59 (4):1481–1509.

Borovička, Jaroslav and Lars Peter Hansen. 2016. Term Structure of Uncertainty in the Macroeconomy. In *Handbook of Macroeconomics*, vol. 2, edited by John Taylor and Harald Uhlig, 1641–1696. Elsevier B.V.

Borovička, Jaroslav, Lars Peter Hansen, and Jose A. Scheinkman. 2014. Shock Elasticities and Impulse Responses. *Mathematics and Financial Economics* 8.

Borovička, Jaroslav and Lars Peter Hansen. 2014. Examining Macroeconomic Models through the Lens of Asset Pricing. *Journal of Econometrics* 183 (1):67–90.

Bucklew, James A. 2004. *Large Deviation Theory*. Springer Series in. Statistics. New York: Springer Verlag.

Epstein, Larry G. and Stanley E. Zin. 1989. Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57:937–969.

Frisch, R. 1933. *Propagation Problems and Impulse Problems in Dynamic Economics*, 171–205. Allen and Unwin.

Hansen, Lars Peter and Thomas J. Sargent. 1995. Discounted Linear Exponential Quadratic Gaussian Control. *IEEE Transactions on Automatic Control* .

———. 2001. Robust Control and Model Uncertainty. *The American Economic Review* 91:60–66.

———. 2021. Macroeconomic uncertainty prices when beliefs are tenuous. *Journal of Econometrics* 223:222–250.

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long-Run Risk. *Journal of Political Economy* 116:260–302.

Holmes, M.H. 2012. *Introduction to Perturbation Methods*. Texts in Applied Mathematics. Springer New York.

Jacobson, David H. 1973. Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Differential Games. *IEEE Transactions for Automatic Control* AC-18:1124–1131.

Judd, Kenneth L. 1998. *Numerical Methods in Economics*. MIT Press.

Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher A. Sims. 2008. Calculating and using second-order accurate solutions of discrete time dynamic equilibrium models. *Journal of Economic Dynamics and Control* 32:3397–3414.

Kreps, David M. and Evan L. Porteus. 1978. Temporal Resolution of Uncertainty and Dynamic Choice. *Econometrica* 46:185–200.

Lombardo, Giovanni and Harald Uhlig. 2018. A Theory of Pruning. *International Economic Review* 59 (4):1825–1836.

Pohl, Walter, Karl Schmedders, and Ole Wilms. 2018. Higher Order Effects in Asset Pricing Models with Long-Run Risks. *Journal of Finance* 73:1061–1111.

Restoy, Fernando and Philippe Weil. 2011. Approximate equilibrium asset prices. *Review of Finance* 15:1–28.

Schmitt-Grohé, Stephanie and Martín Uribe. 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control* 28 (4):755–775.

Tallarini, Jr., Thomas D. 2000. Risk-sensitive Real Business cycles. *Journal of Monetary Economics* 45:507–532.

Whittle, Peter. 1981. Risk Sensitive Linear Quadratic Gaussian Control. *Advances in Applied Probability* 13 (4):764–777.