Robust Identification of Investor Beliefs

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Motivation

Behavioral “distortions” and “ambiguity aversion” are compelling in environments for which uncertainty is complex and speculation about the future is challenging

- WHAT?
  - We propose and justify a data and model-based method for deducing market beliefs
  - We construct bounds on expectations of unknown future aggregates captured as a nonlinear expectation
- WHY?
  - They provide a formal way to address the public and private sector interest in market perceptions
  - They serve as a diagnostic for models in which asset prices are represented with distorted beliefs

Two observations

Asset prices are:

- ▷ REVEALING: forward-looking and serve as barometers for market beliefs
- ▷ CHALLENGING:
  - ○ entangle beliefs and risk aversion
  - ○ data are sparse along some important dimensions

Two approaches

We could:

- impose rational expectations and explore "exotic" or "ad hoc" models with time-varying risk aversion
- model beliefs that are distorted (relative to rational expectations) justified by a) psychology or b) ambiguity aversion with moderate risk aversion

We speak to this second approach:

> *We bound private sector beliefs by limiting how much these beliefs conflict with the probabilities implied by historical evidence.*

Our method

- presume that a dynamic model is misspecified under rational expectations
- correct this misspecification, we allow for beliefs to differ and to be "distorted" (from rational expectations)
- limit the alternative probabilities using statistical measures of "divergence" that capture the magnitude of the distortion
- derive bounds on the beliefs that are consistent with the observed asset prices and survey evidence

Basic formulation

- Moment equations under rational expectations:

$$\mathbb{E}\left[f(X,\theta) \mid \mathfrak{A}\right] = 0.$$

where the function f captures the parameter dependence (θ) along with variables (X) observed by the econometrician.

- A typical asset pricing example:

$$\mathbb{E}(SRet - \mathbf{1}_n | \mathfrak{A}) = 0$$

where Ret is a vector of returns, S is the stochastic discount factor (SDF), $\mathfrak{I}$ denote the investor information set.

For simplicity, I will drop the parameter dependence but comment later on how unknown parameters can be included.

Market beliefs

We consider conditional moment restrictions of the form:

$$\widetilde{\mathbb{E}}\left[f(X) \mid \mathfrak{A}\right] = \mathbb{E}\left[Nf(X) \mid \mathfrak{A}\right] = 0.$$

where $N \geq 0$ and $\mathbb{E}\left(N \mid \mathfrak{A}\right) = 1$.

The random variable N provides a flexible change in the probability measure. N captures how the rational expectations are altered by market beliefs.

- ▷ each N is a "belief distortion"
- ▷ N not uniquely identified!

General applicability to dynamic, stochastic, general equilibrium models.

Two Applications

- ▷ long-term risk-neutral pricing

$$S = (Ret^h)^{-1}$$

  where Ret^h is the limiting holding period return on a long-term bond

- ▷ unitary relative risk aversion in recursive utility

$$S = (Ret^w)^{-1}$$

  where Ret^w is the one-period return on the wealth portfolio

Digression 1

For recursive utility,

- The SDF ratio is:

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1} \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}.$$

- The return on wealth is:

$$Ret^w_{t+1} = \beta^{-1} \left(\frac{V_{t+1}}{R_t}\right)^{1-\rho} \left(\frac{C_{t+1}}{C_t}\right)^{\rho}.$$

Observation: for $\gamma = 1$, $\left(Ret^w_{t+1}\right)^{-1}$ is the one-period stochastic discount factor ratio

Digression 2

For long-term approximation, consider the eigenvalue problem:

$$\mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right) e^s(X_{t+1}) \mid \mathfrak{A}_t\right] = \exp(\eta^s) e^s(X_t)$$

- ▷ Limiting holding-period return is

$$Ret^h_{t+1} = \exp(-\eta^s)\left[\frac{e^s(X_{t+1})}{e^s(X_t)}\right]$$

- ▷ One-period transition for the long-term risk neutral probability:

$$N_{t+1} = \left(\frac{S_{t+1}}{S_t}\right)\left[\frac{e^s(X_{t+1})}{e^s(X_t)}\right]\exp(-\eta^s)$$

Observation: $\frac{S_{t+1}}{S_t} = N_{t+1}\left(R^h_{t+1}\right)^{-1}$.

Proportional risk premia

The proportional risk premia from the perspective of the altered probability is:

$$\log \mathbb{E}\left(NRet \mid \mathfrak{A}\right) + \mathbf{1}_n \log \mathbb{E}\left(NS \mid \mathfrak{A}\right).$$

- The first term is the logarithm altered expectation of *Ret*
- The second term is the negative of the logarithm of the risk-free return

Our methods allow us to compare the rational expectations version of the risk compensations to bounds on these proportional compensations as implied by market data.

Dynamic recursive formulation

- environment: Baseline probability triple $(\Omega, \mathfrak{G}, \mathsf{P})$ used to govern the data generation
- alternative probability measure Q
- conditioning information: let $\mathfrak{A}_t$ denote the date t information (sigma algebra) where $\mathfrak{A}_t \subset \mathfrak{A}_{t+1}$

Recall: Q_t and P_t are the restrictions of Q and P to $\mathfrak{A}_t$

Alternative probabilities

- consider probabilities Q for which there exists an $N = \{N_{t+1} : t = 0, 1, ...\} \geq 0$ where N_{t+1} is in the date $t+1$ information set and satisfies:

$$\int B_t d\mathsf{Q}_t = \int \mathbb{E}\left(N_{t+1} B_{t+1} \mid \mathfrak{A}_t\right) d\mathsf{Q}_t$$

  for bounded stochastic process B
- form

$$M_T = \prod_{t=1}^{T} N_t$$

  where $\mathbb{E}\left(M_T B_T \mid \mathfrak{A}_0\right)$ is the conditional expectation of B_T under Q.

Note: N_{t+1} distorts the one-period transition probabilities between dates t and $t+1$.

Conditional Divergence

We use the conditional version of ϕ divergence as an important building block:

$$\mathbb{E}\left[\phi(N_{t+1}) \mid \mathfrak{A}_t\right]$$

for a strictly convex function ϕ defined on $(0, \infty)$ with $\phi(1) = 0$.

- by Jensen's Inequality,

$$\mathbb{E}\left[\phi(N_{t+1}) \mid \mathfrak{A}_t\right] \geq 0.$$

- leading example:

$$\phi(n) = n \log n$$

  which is conditional relative entropy or Kullback-Leibler devergence.

Equivalent representation

Construct a function ψ such that

$$n\psi\left(\frac{1}{n}\right) = \phi(n).$$

Observations:

- ψ is also strictly convex with $\psi(1) = 0$
- ψ satisfies

$$\mathbb{E}\left[N_{t+1}\psi\left(\frac{1}{N_{t+1}}\right) \mid \mathfrak{A}_t\right] = \mathbb{E}\left[\phi(N_{t+1}) \mid \mathfrak{A}_t\right]$$

- for Kullback-Leibler

$$\psi(n) = -\log n$$

Intertemporal divergence

$$\mathcal{R}(N) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[M_T \sum_{t=1}^{T} \psi\left(\frac{1}{N_t}\right) \mid \mathfrak{A}_0\right]$$

Observations:

- by the Law of Large Numbers for stationary, ergodic processes:

$$\int \psi\left(\frac{1}{N_{t+1}}\right) d\mathsf{Q}_{t+1} = \int \mathbb{E}\left[\phi(N_{t+1}) \mid \mathfrak{A}_t\right] d\mathsf{Q}_t.$$

  where Q is the probability measure implied by N.
- Divergence depends on N_{t+1} and Q_t which are linked via the stationarity restriction.

Problem of interest

For a given function g, we solve:

$$\inf_{N \geq 0} \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left[M_T \sum_{t=1}^{T} g(X_t) \middle| \mathfrak{A}_0 \right]$$

subject to the constraints

$$\begin{aligned} \mathcal{R}(N) &\leq \kappa \\ \mathbb{E}[N_{t+1} f(X_{t+1}) \mid \mathfrak{I}_t] &= 0 \\ \mathbb{E}[N_{t+1} \mid \mathfrak{I}_t] &= 1 \\ M_{t+1} &= N_{t+1} M_t \end{aligned}$$

Impose additional restrictions to ensure implied probability measures satisfy the Law of Large Numbers.

Solution

Three steps:

i) introduce a nonnegative multiplier to enforce the constraint $\mathcal{R}(N) \leq \kappa$ and solve the problem for alternative values of this multiplier,

ii) use a martingale decomposition of the objective to produce a recursive representation of the multiplier problem,

iii) solve this problem using recursive methods familiar from dynamic programming.

Step one

Solve

$$\inf_{N \geq 0} \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left(M_T \left[\sum_{t=1}^{T} g(X_t) + \xi \psi \left(\frac{1}{N_t} \right) \right] \Big| \mathfrak{A}_0 \right) - \xi \kappa$$

subject to the constraints

$$\begin{aligned} \mathbb{E}[N_{t+1} f(X_{t+1}) \mid \mathfrak{I}_t] &= 0 \\ \mathbb{E}[N_{t+1} \mid \mathfrak{I}_t] &= 1 \\ M_{t+1} &= N_{t+1} M_t \end{aligned}$$

where $\xi \geq 0$ is a Lagrange multiplier

Taking the supremum over ξ enforces the divergence constraint.

Martingale decomposition

Recursion: find a real number μ and a stochastic process v such that

$$\mathbb{E}\left(N_{t+1}\left[g(X_{t+1})+\xi\psi\left(\frac{1}{N_{t+1}}\right)+v_{t+1}\right] \mid \mathfrak{A}_t\right)-\mu-v_t=0$$

Observe that

$$\sum_{t=1}^{T}\left[g(X_{t+1})+\xi\psi\left(\frac{1}{N_{t+1}}\right)\right]-T\mu+v_T-v_0$$

is martingale under $\mathbb{Q}$.

Step three

Recursion: find a number μ and a process v such that

$$\inf_{N_{t+1} \geq 0} \mathbb{E}\left(N_{t+1}\left[g(X_{t+1}) + \xi\psi\left(\frac{1}{N_{t+1}}\right) + v_{t+1}\right] \mid \mathfrak{A}_t \right) - \mu - v_t = 0$$

subject to

$$\begin{aligned} \mathbb{E}\left[N_{t+1} f(X_{t+1}) \mid \mathfrak{A}_t\right] &= 0 \\ \mathbb{E}\left(N_{t+1} \mid \mathfrak{A}_t\right) &= 1 \end{aligned}$$

Compute using dynamic programming methods.

Nonlinear expectation

We represent restricted belief distortions by an alternative nonlinear expectation. $\mathbb{K}$ maps bounded functions g into real numbers and satisfies:

i) if $g_2 \geq g_1$, then $\mathbb{K}(g_2) \geq \mathbb{K}(g_1)$.

ii) if g constant, then $\mathbb{K}(g) = g$.

iii) $\mathbb{K}(\mathsf{r}g) = \mathsf{r}\mathbb{K}(g), \quad \mathsf{r} \geq 0$

iv) $\mathbb{K}(g_1) + \mathbb{K}(g_2) \leq \mathbb{K}(g_1 + g_2)$

Unitary risk aversion

- consider the recursive utility model as in Kreps and Porteus and Epstein and Zin
- explore belief distortions instead of large and/or time-varying risk aversion
- value assets with the stochastic discount factor

$$S_{t+1} = N_{t+1}(Ret^{w}_{t+1})^{-1}$$

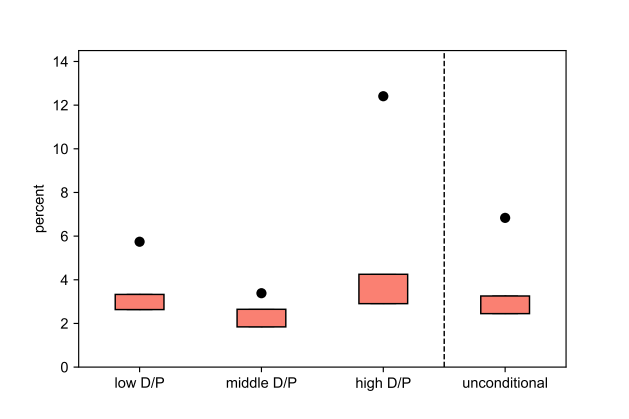
  where Ret^{w}_{t+1} denotes the return on wealth.
- Expected logarithm of the wealth portfolio:

$$E\left(N_{t+1} \log Ret^{w}_{t+1} \mid \mathfrak{A}_t\right) = -\log\beta + \rho E\left[N_{t+1}\left(\widehat{C}_{t+1} - \widehat{C}_t\right) \mid \mathfrak{A}_t\right].$$

  since $\widehat{R}_t = \mathbb{E}\left(N_{t+1}\widehat{V}_{t+1} \mid \mathfrak{A}_t\right)$.

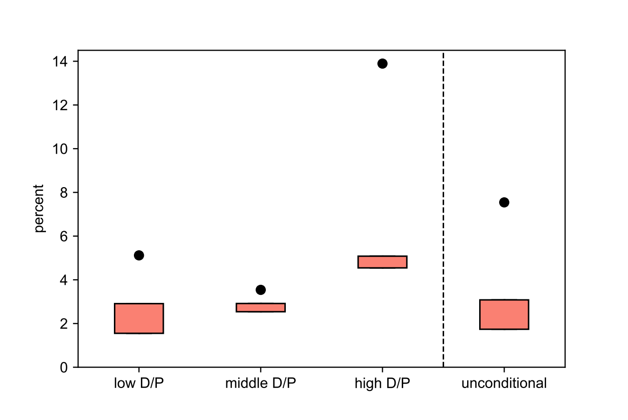
Simple link between the expected log return on wealth and expected growth rate in the macro economy.

Risk compensation



The ·'s are empirical averages and the boxes give the imputed bounds when we inflated the minimum relative entropy by 20%. The minimum relative entropy is .028 with a half-life of 24 quarters.

Expected log market return



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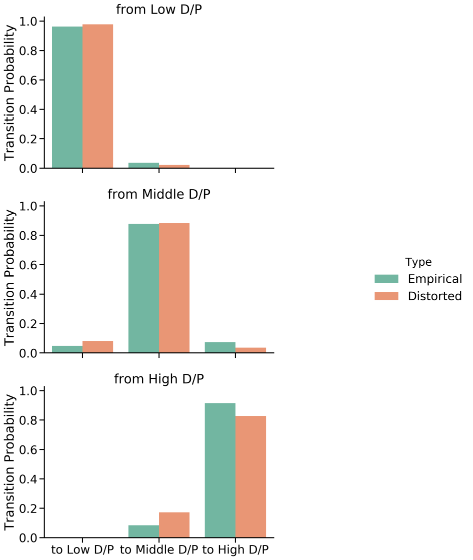
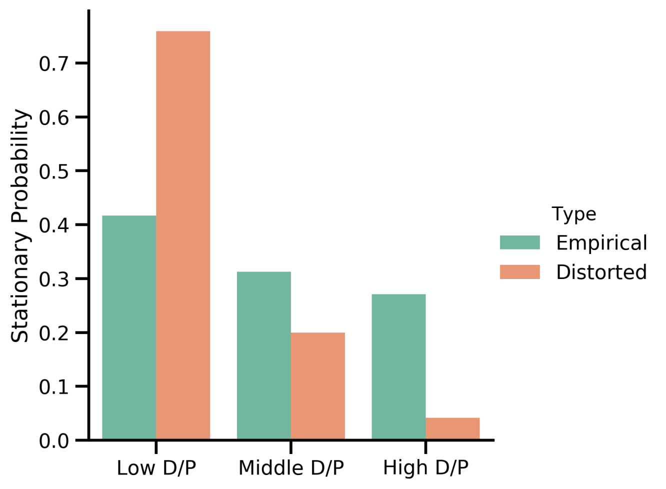
Transition Probability



Stationary Distribution



Concluding Remarks

Use intertemporal statistical divergence as a form of bounded rationality - private sector belief distortions are more prominent when statistical inference challenges are more difficult.

Extensions:

- incorporate parameter dependence in f and g by including an additional minimization over the parameter space
- bound ratios (conditional expectations), log differences (risk compensations), etc with extra one-dimensional minimizations
- incorporate into policy problem where the policymaker cares about the beliefs of private sector
- provide statistical inference methods for our bound measurements