Some Continuous-time Limits

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Continuous time

Level evolution:

$$dV_t = V_t \mu_t^V dt + V_t \sigma_t^V \cdot dW_t$$

Logarithm evolution:

$$d\widehat{V}_t = \hat{\mu}_t^V dt + \sigma_t^V \cdot dW_t$$

where $\hat{\mu}_t^V = \mu_t^V - \frac{1}{2}|\sigma_t^V|^2$.

Discrete-time approximation

$$\widehat{V}_t = \frac{1}{1-\rho} \log \left[(1-\beta_\epsilon) \exp \left[(1-\rho) \widehat{C}_t \right] + \beta_\epsilon \exp \left[(1-\rho) \widehat{R}_t \right] \right]$$
$$\widehat{R}_t = \frac{1}{1-\gamma} \log \mathbb{E} \left[\exp \left((1-\gamma) \widehat{V}_{t+\epsilon} \right) \mid \mathfrak{A}_t \right]$$

where $\beta_\epsilon = \exp(-\delta\epsilon)$. Rewrite this as:

$$\frac{1}{1-\rho} \log \left[(1-\beta_\epsilon) \exp \left[(1-\rho) \left(\widehat{C}_t - \widehat{V}_t \right) \right] + \beta_\epsilon \exp \left[(1-\rho) \left(\widehat{R}_t - \widehat{V}_t \right) \right] \right]$$
$$= 0$$
$$\widehat{R}_t - \widehat{V}_t = \frac{1}{1-\gamma} \log \mathbb{E} \left(\exp \left[(1-\gamma) \left(\widehat{V}_{t+\epsilon} - \widehat{V}_t \right) \right] \mid \mathfrak{A}_t \right)$$

Discrete-time approximation cont.

$$
\begin{aligned}
\frac{d}{d\epsilon}\left(\widehat{R}_t - \widehat{V}_t\right)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\,\frac{1}{1-\gamma}\log\mathbb{E}\left(\exp\left[(1-\gamma)\left(\widehat{V}_{t+\epsilon} - \widehat{V}_t\right)\right] \mid \mathfrak{A}_t\right)\Big|_{\epsilon=0} \\
&= \hat{\mu}_t^V + \frac{(1-\gamma)}{2}|\sigma_t^V|^2 \\
&= \mu_t^V - \frac{\gamma}{2}|\sigma_t^V|^2
\end{aligned}
$$

by local log normality.

Discrete-time approximation cont.

Differentiate with respect to ϵ:

$$\frac{1}{1-\rho}\log\left[(1-\beta_\epsilon)\exp\left[(1-\rho)\left(\widehat{C}_t-\widehat{V}_t\right)\right]+\beta_\epsilon\exp\left[(1-\rho)\left(\widehat{R}_t-\widehat{V}_t\right)\right]\right]$$
$$=0$$

$$\begin{aligned}
&\frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho}-1\right]}{1-\rho}+\frac{d}{d\epsilon}\left(\widehat{R}_t-\widehat{V}_t\right)\Big|_{\epsilon=0}\\
&=\frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho}-1\right]}{1-\rho}+\mu_t^V-\frac{\gamma}{2}|\sigma_t^V|^2\\
&=0.
\end{aligned}$$

Duffie-Epstein refer to γ as a "variance multiplier."

Robustness and subjective beliefs

Girsanov theory:

- Consider H processes with the same dimension as the underlying Brownian motion.
- Form a positive martingales to induce changes in the probability measure:

$$dM_t = M_t H_t \cdot dW_t$$

- Observe that

$$d \log M_t = -\frac{1}{2} |H_t|^2 dt + H_t \cdot dW_t$$

- Under the H change of measure:

$$dW_t = H_t dt + d\widetilde{W}_t^H$$

where $\widetilde{W}^H$ is a Brownian motion.

Robustness and subjective beliefs

Recall two relations:

$$
\begin{aligned}
d \log M_t &= -\frac{1}{2} |H_t|^2 \, dt + H_t \cdot dW_t \\
dW_t &= H_t dt + d\widetilde{W}_t^H
\end{aligned}
$$

Observations:

- Change of measure induces a local mean or drift H process in the Brownian motion
- Under the change of measure, the drift in $\log M$ is $\frac{1}{2} |H_t|^2$ - local relative entropy

A heuristic digression

Discrete approximation:

$$\log M_{t+\epsilon} - \log M_t = -\frac{\epsilon}{2} |H_t|^2 + H_t \cdot (W_{t+\epsilon} - W_t)$$

Let w be a realized $W_{t+\epsilon} - W_t$ and h be a realization H_t. Then $\log M_{t+\epsilon} - \log M_t$ contributes $-\frac{\epsilon}{2} h'h + h \cdot w$ to the log-likelihood. The standard normal density for $(W_{t+\epsilon} - W_t)$ contributes $-\frac{1}{\epsilon} w'w$. Put together, we have a log-likelihood:

$$-\frac{\epsilon}{2} h'h + h \cdot w - \frac{1}{2\epsilon} w'w = -\frac{1}{2\epsilon} (w - \epsilon h)'(w - \epsilon h)$$

The altered density has mean ϵh, which is the realized value of H_t.

Robust adjustment in cont. time

- Replace the value function drift $\hat{\mu}_t^V$ with

$$\hat{\mu}_t^V + \sigma_t^V \cdot H_t$$

- Solve:

$$\min_{H_t} \hat{\mu}_t^V + \sigma_t^V \cdot H_t + \frac{\xi_m}{2} H_t \cdot H_t$$

- Minimizing H_t:

$$H_t^* = -\frac{1}{\xi_m} \sigma_t^V$$

- Minimized objective

$$\hat{\mu}_t^V - \frac{1}{\xi_m} \left| \sigma_t^V \right|^2$$

Set $\gamma - 1 = \frac{1}{2\xi_m}$ to connect to our previous analysis.

Smooth ambiguity/prior robustness 1

- Suppose that
$$d\widehat{V}_t = \hat{\mu}_t^V(\theta)dt + \sigma_t^V dW_t$$
where θ is not observed.
- Replace:
$$\frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho} - 1\right]}{1-\rho} + \hat{\mu}_t^V - \frac{1}{2\xi_m}\,|\sigma_t^V|^2 = 0$$
with
$$\frac{\delta\left[\left(\frac{C_t}{V_t}\right)^{1-\rho} - 1\right]}{1-\rho} - \frac{1}{\xi_p}\log\mathbb{E}\left(\exp\left[-\xi_p\hat{\mu}_t^V(\theta)\right] \mid \mathfrak{A}_t\right) - \frac{1}{2\xi_m}|\sigma_t^V|^2 = 0$$

Hansen-Maio (PNAS)

Smooth ambiguity/prior robustness 2

$$
\begin{aligned}
& -\frac{1}{\xi_p} \log \mathbb{E} \left(\exp \left[-\xi_p \hat{\mu}_t^V(\theta) \right] \mid \mathfrak{A}_t \right) \\
& = \min_{N_t(\theta)} \mathbb{E} \left[N_t(\theta) \hat{\mu}_t^V(\theta) \mid \mathfrak{A}_t \right] + \xi_p \mathbb{E} \left[N_t(\theta) \log N_t(\theta) \mid \mathfrak{A}_t \right]
\end{aligned}
$$

where

$$
N_t^*(\theta) = \frac{\exp \left[-\xi_p \hat{\mu}_t^V(\theta) \right]}{\mathbb{E} \left(\exp \left[-\xi_p \hat{\mu}_t^V(\theta) \right] \mid \mathfrak{A}_t \right)}
$$

Exponential tilting

Value functions in a Markov world

- Markov state dynamics

$$dX_t = \mu_x(X_t) + \sigma_x(X_t)dW_t$$

- Suppose that

$$\widehat{V}_t = f(X_t)$$

- Then

$$\hat{\mu}_t^V = \frac{\partial f}{\partial x}(X_t)\mu_x(X_t) + \frac{1}{2}\text{trace}\left[\sigma_x(X_t)'\frac{\partial^2 f}{\partial x \partial x'}(X_t)\sigma_x(X_t)\right]$$

$$\sigma_t^V = \sigma_x(X_t)'\left[\frac{\partial f}{\partial x}(X_t)\right]'$$

$$\left|\sigma_t^V\right|^2 = \left[\frac{\partial f}{\partial x}(X_t)\right]\sigma_x(X_t)\sigma_x(X_t)'\left[\frac{\partial f}{\partial x}(X_t)\right]'$$

Note that $\frac{\partial f}{\partial x}$ is n (# of state variables) by 1 and $\frac{\partial^2 f}{\partial x \partial x'}$ is n by n.

Stochastic discount factor evolution

Abstract from ambiguity (see Hansen-Miao, forthcoming ET for the inclusion of ambiguity)

- Local evolution:

$$
\begin{aligned}
dS_t &= S_t \mu_t^S dt + S_t \sigma_t^S \cdot dW_t \\
d\log S_t &= \hat{\mu}_t^S dt - \frac{1}{2}\left|\sigma_t^S\right|^2 dt + \sigma_t^V \cdot dW_t \\
dC_t &= C_t \mu_t^C dt + C_t \sigma_t^C \cdot dW_t \\
d\log C_t &= \hat{\mu}_t^C dt + \sigma_t^C \cdot dW_t
\end{aligned}
$$

  where

$$
\begin{aligned}
\hat{\mu}_t^S &= \mu_t^S - \frac{1}{2}\left|\sigma_t^S\right|^2 \\
\hat{\mu}_t^C &= \mu_t^C - \frac{1}{2}\left|\sigma_t^C\right|^2
\end{aligned}
$$

- $-\mu_t^S$ local risk-free rate; $-\sigma_t^S$ -vector of local shock prices

Evolution of logarithms

- Evolution over a period of length ϵ :

$$\log S_{t+\epsilon} - \log S_t = -\epsilon\delta - \rho\left[\widehat{C}_{t+\epsilon} - \widehat{C}_t\right] \\ + (1-\gamma)\left[\widehat{V}_{t+\epsilon} - \widehat{R}_t\right] + (\rho - 1)\left[\widehat{V}_{t+\epsilon} - \widehat{R}_t\right]$$

- Instantaneous counterparts

$$\begin{aligned} \hat{\mu}_t^S &= -\delta - \rho\hat{\mu}_t^C - \frac{(1-\gamma)^2}{2}\left|\sigma_t^V\right|^2 - \frac{(1-\gamma)(\rho-1)}{2}\left|\sigma_t^V\right|^2 \\ &= -\delta - \rho\hat{\mu}_t^C + \frac{(\gamma-1)(\rho-\gamma)}{2}\left|\sigma_t^V\right|^2 \\ \sigma_t^S &= -\rho\sigma_t^C + (1-\gamma)\sigma_t^V + (\rho-1)\sigma_t^V \\ &= -\rho\sigma_t^C + (\rho-\gamma)\sigma_t^V \end{aligned}$$

- $\mu_t^S = \hat{\mu}_t^S + \frac{1}{2}\left|\sigma_t^S\right|^2$

Elaboration

Consider:

$$(1-\gamma)\left[\widehat{V}_{t+\epsilon} - \widehat{R}_t\right]$$

Recall that

$$\mathbb{E}\left(\exp\left[(1-\gamma)\left[\widehat{V}_{t+\epsilon} - \widehat{R}_t\right]\right] \mid \mathfrak{A}_t\right) = 1.$$

Thus

$$\lim_{\epsilon \downarrow 0} \frac{(1-\gamma)}{\epsilon}\mathbb{E}\left(\widehat{V}_{t+\epsilon} - \widehat{R}_t \mid \mathfrak{A}_t\right) + \frac{(1-\gamma)^2}{2}\left|\sigma_t^V\right|^2 = 0$$

and

$$\lim_{\epsilon \downarrow 0} \frac{(\rho-1)}{\epsilon}\mathbb{E}\left(\widehat{V}_{t+\epsilon} - \widehat{R}_t \mid \mathfrak{A}_t\right) + \frac{(1-\gamma)(\rho-1)}{2}(\left|\sigma_t^V\right|^2 = 0.$$