Production and Asset Pricing

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AK adjustment cost economy

- Output
$$C_t + I_t = \alpha K_t$$
with multiplier denoted MC_t. Divide by K_t to obtain:
$$\frac{C_t}{K_t} + \frac{I_t}{K_t} = \alpha.$$
- Capital evolution:
$$K_{t+1} = K_t \left[1 + \theta_2 \left(\frac{I_t}{K_t}\right)\right]^{\theta_1} \exp(B_{t+1} - B_t)$$
with costate MK_{t+1} where $0 < \theta_1 < 1, \theta_1\theta_2 = 1$.

Exogenous state evolution

$$
\begin{aligned}
B_{t+1} - B_t &= -\delta_k + Z_t - \frac{1}{2} \mid \sigma_k \mid^2 + \sigma_k \cdot W_{t+1} \\
Z_{t+1} &= \mathbb{A} Z_t + \mathbb{B} W_{t+1}
\end{aligned}
$$

Adjustment cost specification

Functional form:

$$(1 + \theta_2 i)^{\theta_1}$$

where $0 < \theta_1 < 1$ and $\theta_1 \theta_2 = 1$.

Second-order approximation (around $i = 0$)

$$(1 + \theta_2 i)^{\theta_1} \approx 1 + i + \frac{(1 - \theta_2)}{2} i^2$$

State equation rewritten

$$\left(\frac{K_{t+1}}{K_t}\right) = \left[1 + \theta_2 \left(\frac{I_t}{K_t}\right)\right]^{\theta_1} \exp(B_{t+1} - B_t)$$

or

$$\log K_{t+1} - \log K_t = \theta_1 \log\left[1 + \theta_2 \left(\frac{I_t}{K_t}\right)\right] + B_{t+1} - B_t$$

Costate

$$
\begin{aligned}
&MC_t \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right) \mid \mathfrak{A}_t\right] \\
&- MC_t \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right)\left(\left[1+\theta_2\left(\frac{I_t}{K_t}\right)\right]^{-1}\left(\frac{I_t}{K_t}\right)\right] \mid \mathfrak{A}_t\right] \\
&- MK_t + \alpha MC_t = 0
\end{aligned}
$$

Costate rewritten

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right) \mid \mathfrak{A}_t\right] \\
&- \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right)\left(\left[1+\theta_2\left(\frac{I_t}{K_t}\right)\right]^{-1}\left(\frac{I_t}{K_t}\right)\right] \mid \mathfrak{A}_t\right] \\
&- \frac{MK_t}{MC_t} + \alpha = 0
\end{aligned}
$$

A digression

- Solve the optimization problem from the perspective of time 0 while looking ahead to the subsequent time periods t and $t+1$.
- Use conditional expectations as a convenient way to sum across states that are yet to realized.
- The state equation for dates t and $t+1$ with multipliers ℓ_t and ℓ_{t+1} and the output equation with multiplier $\hat{\ell}_t$ add three terms to the Lagrangian:

$$
\begin{aligned}
&\mathbb{E}\left[\ell_t\left(K_t - K_{t-1}\left[1+\theta_2\left(\frac{I_{t-1}}{K_{t-1}}\right)\right]^{\theta_1}\exp(B_t - B_{t-1})\right) \mid \mathfrak{A}_0\right] \\
&+\mathbb{E}\left[\ell_{t+1}\left(K_{t+1} - K_t\left[1+\theta_2\left(\frac{I_t}{K_t}\right)\right]^{\theta_1}\exp(B_{t+1} - B_t)\right) \mid \mathfrak{A}_0\right] \\
&+\mathbb{E}\left[\hat{\ell}_t\left(C_t + I_t - \alpha K_t\right) \mid \mathfrak{A}_0\right]
\end{aligned}
$$

Digression continued

The optimization problem has a recursive structure, leading us to perform calculations conditioned on time t leading us to look at:

$$\ell_t \left(K_t - K_{t-1} \left[1 + \theta_2 \left(\frac{I_{t-1}}{K_{t-1}} \right) \right]^{\theta_1} \exp(B_t - B_{t-1}) \right)$$
$$+ \mathbb{E} \left[\ell_{t+1} \left(K_{t+1} - K_t \left[1 + \theta_2 \left(\frac{I_t}{K_t} \right) \right]^{\theta_1} \exp(B_{t+1} - B_t) \right) \mid \mathfrak{A}_t \right]$$
$$+ \hat{\ell}_t \left(C_t + I_t - \alpha K_t \right)$$

The first-order conditions for consumption imply that

$$\hat{\ell}_t = MC_t$$

We then construct MK_t and MK_{t+1} using the formulas:

$$\ell_t = MK_t \quad \ell_{t+1} = MK_{t+1} \left(\frac{S_{t+1} MC_t}{S_t MC_{t+1}} \right)$$

Digression continued

Suppose we revert to the date zero perspective. The first-order conditions for consumption now imply that

$$\hat{\ell}_t = MC_t \left(\frac{S_t MC_0}{S_0 MC_t} \right) = \frac{S_t MC_0}{S_0}$$

In this case, we set

$$\begin{aligned}
\ell_t &= MK_t \left(\frac{S_t MC_0}{S_0 MC_t} \right) \\
\ell_{t+1} &= MK_{t+1} \left(\frac{S_{t+1} MC_t}{S_0 MC_{t+1}} \right) \left(\frac{S_t MC_0}{S_0 MC_t} \right) = MK_{t+1} \left(\frac{S_{t+1} MC_0}{S_0 MC_{t+1}} \right)
\end{aligned}$$

to be consistent with the presumed recursive structure.

Digression observations

- The multiplier construction using the ℓ's and $\hat{\ell}$'s depend on the choice of date zero.
- Our derivation rescales the multiple to capture the change in vantage point by exploiting the recursive structure of the problem.
- The solutions for particular ratios of interest are functions of the current Markov state vector

First-order conditions for investment

$$MC_t \mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\frac{MK_{t+1}}{MC_{t+1}}\left(\theta_1\theta_2\left[1+\theta_2\left(\frac{I_t}{K_t}\right)\right]^{\theta_1-1}\exp(B_{t+1}-B_t)\right) \mid \mathfrak{A}_t\right]$$
$$-MC_t = 0.$$

Dividing this first-order condition by MC_t and substituting in $\frac{K_{t+1}}{K_t}$ gives:

$$\mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right)\left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right)\left[1+\theta_2\left(\frac{I_t}{K_t}\right)\right]^{-1} \mid \mathfrak{A}_t\right]$$
$$-1 = 0.$$

A revealing "asset return" formula

$$\mathbb{E}\left[\left(\frac{S_{t+1}}{S_t}\right) R^i_{t+1} | \mathfrak{A}_t\right] - 1 = 0$$

where

$$R^i_{t+1} \stackrel{\text{def}}{=} \left(\frac{MK_{t+1}}{MC_{t+1}}\right)\left(\frac{K_{t+1}}{K_t}\right)\left[1 + \theta_2\left(\frac{I_t}{K_t}\right)\right]^{-1}$$

is the physical return to investment.

Collecting variables and equations

Jump variables: $\log MK_t - \log MC_t, \log I_t - \log K_t, \log C_t - \log K_t$

State variables: $\log K_t - \log K_{t_1}, B_t - B_{t-1}, Z_t$

Three equations in addition the state evolution equations: output equation, costate evolution equation and first-order conditions for investment.

Solve for the jump variables as functions of the state variables.

Observations

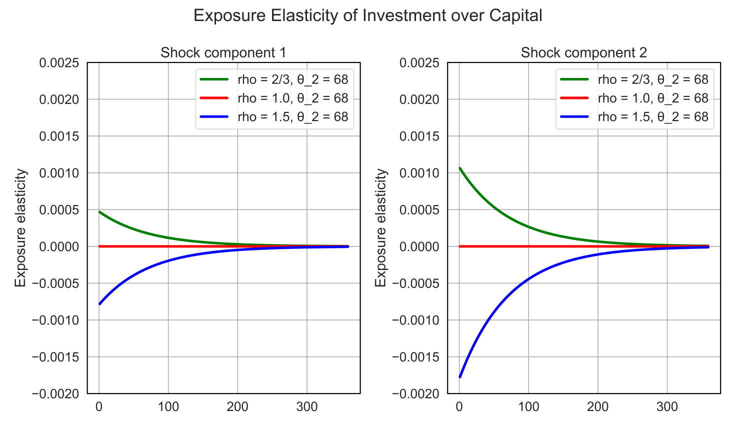
- limited state dependence: jump variable only depend on Z_t
- jump variables are constant when $\rho = 1$, including both the investment capital ratio and the consumption-capital ratio

Recall

$$\log K_{t+1} - \log K_t = \theta_1 \log \left[1 + \theta_2 \left(\frac{I_t}{K_t} \right) \right] + B_{t+1} - B_t$$

Simple connection to an long-run risk type economy when $\rho = 1$.

Exposure elasticity: I over K



Steady states for C/I are .015 for $\rho = 2/3$, .281 for $\rho = 1$ and .510 for $\rho = 3/2$.

Price elasticity: K as growth process