Exploring recursive utility

Lars Peter Hansen (University of Chicago)
Asset Pricing Class
January 2023

Approximating processes

Follow Lombardo and Uhlig (IER, 2018) by considering the stochastic processes indexed by a q.

$$X_{t+1}(\mathsf{q}) = \psi\left[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}\right]$$

- X is an n-dimensional stochastic process
- W is an iid. normally distributed random vector with conditional mean 0 and conditional covariance I.
- $\mathsf{q} = 1$ is the model of interest.

A convenient approximation

Consider a local approximation of the form:

$$X_t \approx X_t^0 + \mathsf{q} X_t^1 + \frac{\mathsf{q}^2}{2} X_t^2$$

where the order zero process is invariant and satisfies:

$$X_{t+1}^0 = \psi\left(X_t^0, 0, 0\right).$$

Order one

The first-derivative process obeys

$$X_{t+1}^{1} = \begin{bmatrix} \psi_{x'}^{1} \\ \psi_{x'}^{2} \\ \vdots \\ \psi_{x'}^{n} \end{bmatrix} X_t^1 + \begin{bmatrix} \psi_{w'}^{1} \\ \psi_{w'}^{2} \\ \vdots \\ \psi_{w'}^{n} \end{bmatrix} W_{t+1} + \begin{bmatrix} \psi_{\mathsf{q}}^{1} \\ \psi_{\mathsf{q}}^{2} \\ \vdots \\ \psi_{\mathsf{q}}^{n} \end{bmatrix}.$$

Write this compactly as a first-order vector autoregression t

$$X_{t+1}^{1} = \psi_{x'} X_t^1 + \psi_{w'} W_{t+1} + \psi_{\mathsf{q}}$$

We assume that the matrix ψ_x' is stable in the sense that all of its eigenvalues are strictly less than one in modulus.

Order two

$$
\begin{aligned}
X_{t+1}^2 = \ & \psi_{x'} X_t^2 + \begin{bmatrix} X_t^{1\prime} \psi_{xx'}^1 X_t^1 \\ X_t^{1\prime} \psi_{xx'}^2 X_t^1 \\ \vdots \\ X_t^{1\prime} \psi_{xx'}^n X_t^1 \end{bmatrix} + 2 \begin{bmatrix} X_t^{1\prime} \psi_{xw'}^1 W_{t+1} \\ X_t^{1\prime} \psi_{xw'}^2 W_{t+1} \\ \vdots \\ X_t^{1\prime} \psi_{xw'}^n W_{t+1} \end{bmatrix} + \begin{bmatrix} W_{t+1}{}' \psi_{ww'}^1 W_{t+1} \\ W_{t+1}{}' \psi_{ww'}^2 W_{t+1} \\ \vdots \\ W_{t+1}{}' \psi_{ww'}^n W_{t+1} \end{bmatrix} \\
& + 2 \begin{bmatrix} \psi_{\mathsf{q}x'}^1 X_t^1 \\ \psi_{\mathsf{q}x'}^2 X_t^1 \\ \vdots \\ \psi_{\mathsf{q}x'}^n X_t^1 \end{bmatrix} + 2 \begin{bmatrix} \psi_{\mathsf{q}w'}^1 W_{t+1} \\ \psi_{\mathsf{q}w'}^2 W_{t+1} \\ \vdots \\ \psi_{\mathsf{q}w'}^n W_{t+1} \end{bmatrix} + \begin{bmatrix} \psi_{\mathsf{qq}}^1 \\ \psi_{\mathsf{qq}}^2 \\ \vdots \\ \psi_{\mathsf{qq}}^n \end{bmatrix}
\end{aligned}
$$

Growth-rate approximation

$$\widehat{Y}_{t+1} - \widehat{Y}_t = \kappa(X_t, \mathsf{q} W_{t+1}, \mathsf{q})$$

Approximate this process by:

$$\widehat{Y}_{t+1} - \widehat{Y}_t \approx \widehat{Y}^0_{t+1} - \widehat{Y}^0_t + \mathsf{q}\left(\widehat{Y}^1_{t+1} - \widehat{Y}^1_t\right) + \frac{\mathsf{q}^2}{2}\left(\widehat{Y}^2_{t+1} - \widehat{Y}^2_t\right)$$

where

$$\begin{aligned}
\widehat{Y}^0_{t+1} - \widehat{Y}^0_t = {} & \kappa(X^0_t, 0, 0) \equiv \eta^y_0 \\
\widehat{Y}^1_{t+1} - \widehat{Y}^1_t = {} & \kappa_{x'} X^1_t + \kappa_{w'} W_{t+1} + \kappa_q \\
\widehat{Y}^2_{t+1} - \widehat{Y}^2_t = {} & \kappa_{x'} X^2_t + {X^1_t}' \kappa_{x,x'} X^1_t + 2{X^1_t}' \kappa_{xw'} W_{t+1} + {W_{t+1}}' \kappa_{ww'} W_{t+1} \\
& + 2\kappa_{q,x'} X^1_t + 2\kappa_{qw'} W_{t+1} + \kappa_{qq}.
\end{aligned}$$

First-order approximation of cont values

$$\widehat{R}_t^1 - \widehat{C}_t^1 = \left(\frac{1}{1-\gamma_o}\right) \log \mathbb{E} \left(\exp \left[(1-\gamma_o) \left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1 \right) \right] \mid \mathfrak{A}_t \right)$$
$$\widehat{V}_t^1 - \widehat{C}_t^1 = \left(\frac{\lambda}{1-\gamma_o}\right) \log \mathbb{E} \left(\exp \left[(1-\gamma_o) \left(\widehat{V}_{t+1}^1 - \widehat{C}_t^1 \right) \right] \mid \mathfrak{A}_t \right)$$

where it is convenient to write

$$\widehat{V}_{t+1}^1 - \widehat{C}_t^1 = \left(\widehat{V}_{t+1}^1 - \widehat{C}_{t+1}^1 \right) + \left(\widehat{C}_{t+1}^1 - \widehat{C}_t^1 \right).$$

and where: $\lambda = \beta \exp \left[(1-\rho) \eta_c^0 \right]$.

Uncertainty measure approximation

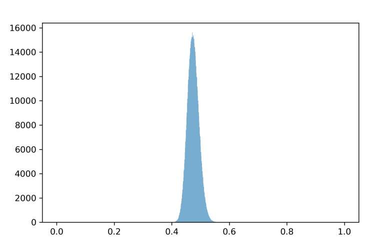
Measure

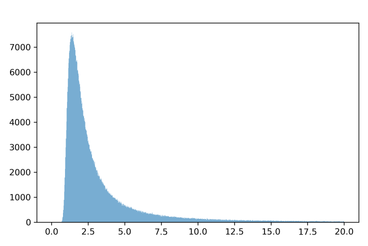
$$N_{t+1}^{*} \stackrel{\text{def}}{=} \left(\frac{V_{t+1}}{R_t} \right)^{1-\gamma}$$

Approximation

$$N_{t+1}^{0} = \frac{\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^{1} - \widehat{C}_{t}^{1}\right)\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left(\widehat{V}_{t+1}^{1} - \widehat{C}_{t}^{1}\right)\right] \mid \mathfrak{A}_t\right)}.$$

Posterior histograms





Left figure: one period volatility and right figure martingale increment volatility for consumption

Second-order approximation of cont values

$$
\begin{aligned}
\widehat{R}_t^2 - \widehat{C}_t^2 &= \mathbb{E}\left[N_{t+1}^0\left(\widehat{V}_{t+1}^2 - \widehat{C}_t^2\right) \mid \mathfrak{A}_t\right] \\
\widehat{V}_t^2 - \widehat{C}_t^2 &= \lambda\mathbb{E}\left[N_{t+1}^0\left(\widehat{V}_{t+1}^2 - \widehat{C}_t^2\right) \mid \mathfrak{A}_t\right) \\
&\quad + (1-\rho)(1-\lambda)\lambda\left(\widehat{R}_t^1 - \widehat{C}_t^1\right)^2
\end{aligned}
$$

SDF approximation

$$\frac{S_{t+1}}{S_t} = N^*_{t+1} \exp\left(\widehat{S}_{t+1} - \widehat{S}_t\right)$$

where

$$N^*_{t+1} \overset{\text{def}}{=} \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}$$

and

$$\widehat{S}_{t+1} - \widehat{S}_t \overset{\text{def}}{=} \log\beta - \rho\left(\widehat{C}_{t+1} + \widehat{C}_t\right) + (\rho - 1)\left(\widehat{V}_{t+1} - \widehat{R}_t\right)$$

SDF approx I

$\widehat{S}_{t+1} - \widehat{S}_t$:

$$\begin{aligned}\log S_{t+1} - \log S_t \approx (1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right] \\ + \left(\widehat{S}^0_{t+1} - \widehat{S}^0_t\right) + \left(\widehat{S}^1_{t+1} - \widehat{S}^1_t\right) + \frac{1}{2}\left(\widehat{S}^2_{t+1} - \widehat{S}^2_t\right)\end{aligned}$$

where $\log N^*_{t+1} \approx \log \widetilde{N}_{t+1}$ and

$$\widetilde{N}_{t+1} \overset{\text{def}}{=} \exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right]$$

Observation:

- approximation $\widetilde{N}_{t+1}$ does not have a conditional expectation that is equal to one

SDF approx II

$$\log \widetilde{N}_{t+1} = \frac{\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right]}{\mathbb{E}\left(\exp\left[(1-\gamma_o)\left[\left(\widehat{V}^1_{t+1} - \widehat{R}^1_t\right) + \frac{1}{2}\left(\widehat{V}^2_{t+1} - \widehat{R}^2_t\right)\right]\right] \mid \mathfrak{A}_t\right)}.$$

Observation:

- approximation $\widetilde{N}_{t+1}$ has a conditional expectation that is equal to one
- induces a change in probability measure for W_{t+1} with a conditional mean that is affine in X^1_t and an altered state independent covariance matrix

Model solution

$$\mathbb{E}\left[N^*_{t+1} Q^*_{t+1} \psi_1(X_t, J_t, X_{t+1}, J_{t+1}) \mid \mathfrak{A}_t\right] + \psi_2(X_t, J_t) = 0$$

where X_t is a state vector, J_t is jump vector and

- $$N^*_{t+1} \stackrel{\text{def}}{=} \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}$$
- $$Q^*_{t+1} \stackrel{\text{def}}{=} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1}$$
- $\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}$ is absorbed in ψ_1.

Model solution

Schmidtt-Grohe and Lombardo and Uhlig treat the case in which

$$\mathbb{E}\left[\psi_1(X_t, J_t, X_{t+1}, J_{t+1}) \mid \mathfrak{A}_t\right] + \psi_2(X_t, J_t) = 0$$

combined with the state evolution. $m + n$ equations where n is the number of states and m is the number of jump variables.

- First order - affine difference equation in (X_t^1, J_t^1). $n + m$ equations and n initial conditions for the state. Find J_t^1 as an affine function of X_t^1 so that the combined system is stochastically stable.
- Second order - affine difference equation in (X_t^2, J_t^2) and linear-quadratic in (X_t^1, J_t^1). Find J_t^2 as an affine function of X_t^2 and linear-quadratic function of X_t^1 so that the combined system is stochastically stable

Incorporating recursive utility

Iterate to convergence

- Take guesses for N^*_{t+1} and Q^*_{t+1} and use the guess for N^*_{t+1} as a change in probability. Compute first and second order approximations for states and jumps
- Given approximate state dynamics deduce a new guess for N^*_{t+1} and Q^*_{t+1} by applying the approximation formulas