Term Structure of Uncertainty and Pricing

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Ysset Pricing Class
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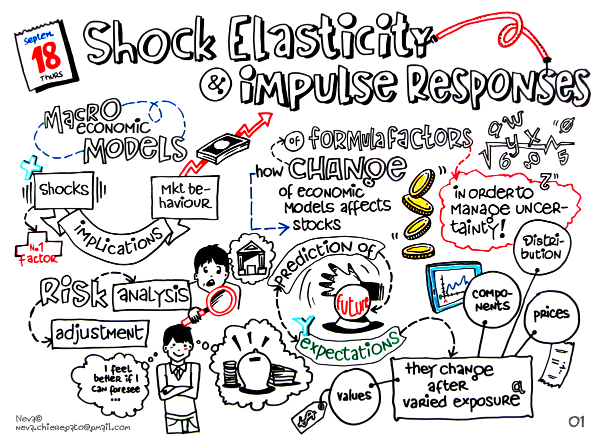
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Shock Elasticity & Impulse Responses
Macro economic Models
Shocks
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Implications
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Risk analysis
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how Change of economic Models affects stocks
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Prediction of
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Underlying environment

- Yn n-dimensional, stationary and ergodic Markov process $X = \{X_t : t = 1, 2, ...\}$
- a k-dimensional process W of independent and identically distributed shocks.
- a date t information set $\mathfrak{A}_t$ generated by the histories of W and X_0.
- a dynamic evolution:

$$X_{t+1} \quad = \quad \psi\left(X_t, W_{t+1}\right)$$

Multiplicative functionals

The (log) increments of a “multiplicative functional’ $\exp(Y)$ satisfy

$$Y_{t+1} - Y_t = \kappa\left(X_t, W_{t+1}\right).$$

Two applications:

- stochastic discount factor process S over alternative horizons
- stochastic cash flow growth G over alternative horizons

Observation: product of multiplicative functionals is a multiplicative functional.

Example

$$Y_{t+1} - Y_t = \kappa \left(X_t, W_{t+1}\right) = \beta \left(X_t\right) + \alpha \left(X_t\right) \cdot W_{t+1}$$

where

- $\beta(x)$ allows for nonlinearity in the conditional mean,
- $\alpha(x)$ introduces stochastic volatility.

Observations

- when X is a vector autoregression, β is affine in x and α is constant, $\exp(Y)$ is log normal.
- sets the stage for continuous-time counterparts

Stochastic discount process

Y stochastic discount factor process S is a positive (with probability one) stochastic process such that for any $t, \tau \geq 0$ and payoff $G_{t+\tau}$ maturing at time $t+\tau$, the time-t price is given by

$$\pi_{t,t+\tau}(G) = \mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t}\right) G_{t+\tau} \mid \mathfrak{A}_t\right] = G_t \mathbb{E}\left[\left(\frac{S_{t+\tau} G_{t+\tau}}{S_t G_t}\right) \mid \mathfrak{A}_t\right]$$

Observation: This construction does not restrict how we initialize the date zero stochastic discount factor, S_0. We may impose $S_0 = 1$ as a normalization.

Proportional risk compensations

$$\frac{1}{\tau}\log\mathbb{E}\left(\frac{G_{t+\tau}}{G_t}\mid\mathfrak{A}_t\right)-\frac{1}{\tau}\log\mathbb{E}\left(\frac{G_{t+\tau}S_{t+\tau}}{G_tS_t}\mid\mathfrak{A}_t\right)+\frac{1}{\tau}\log\mathbb{E}\left(\frac{S_{t+\tau}}{S_t}\mid\mathfrak{A}_t\right)$$

Interpretation:

- ▷ first term: logarithm of the expected cash flow growth
- ▷ first and second term: logarithm of the expected return
- ▷ third term: the negative of the logarithm of the riskless return

all over investment horizon τ.

Shows how risk compensations compound over time!

Long-term Asset Pricing

Factorization

$$\exp(Y_t - Y_0) = \exp(\eta t) \left(\frac{M_t}{M_0}\right) \left[\frac{e(X_0)}{e(X_t)}\right]$$

where

- η is a growth or decay rate
- M is a multiplicative martingale:

$$\mathbb{E}\left[\left(\frac{M_{t+1}}{M_t}\right) \mid \mathfrak{A}_t\right] = 1$$

- $e > 0$ satisfies the eigenvalue problem (formally a Perron-Frobenius problem):

$$\mathbb{E}\left[\exp(Y_{t+1} - Y_t)e(X_{t+1} \mid \mathfrak{A}_t\right] = \eta e(X_t).$$

Conversely, solve the eigenvalue problem and construct the multiplicative martingale M.

Multiplicative martingale

Recall

$$\mathbb{E}\left[\exp(Y_{t+1} - Y_t)e(X_{t+1}) \mid \mathfrak{A}_t\right] = \eta e(X_t),$$

and

$$\frac{M_{t+1}}{M_t} = \exp(-\eta)\exp(Y_{t+1} - Y_t)\left[\frac{e(X_{t+1})}{e(X_t)}\right]$$

- $\frac{M_{t+1}}{M_t}$ induces a change of probability measure where

$$\widetilde{\mathbb{E}}\left(B_{t+1} \mid \mathfrak{A}_t\right) = \mathbb{E}\left[\left(\frac{M_{t+1}}{M_t}\right) B_{t+1} \mid \mathfrak{A}_t\right)$$

  for any bounded random variable B_{t+1} in the date $t+1$ conditioning information set.
- Eigenvalue problem solution is essentially unique if the implied dynamics are stochastically stable. See Hansen-Scheinkman (2009, Econometrica) and Hansen (2012, Econometrica).

Log versus levels

Suppose that

$$Y_t - Y_0 = \underbrace{\hat{\eta} t}_{\text{trend}} + \underbrace{\log \widehat{M}_t}_{\text{martingale}} - \underbrace{f(X_t)}_{\text{stationary}} + \underset{\text{invariant}}{f(X_0)}$$

Observations:

- ▷ the martingales are different
- ▷ whenever one has a martingale component so does the other

Reference: Hansen (2012, Econometrica)

Factoring the stochastic discount factor

$$\frac{S_t}{S_0} = \exp\left(t\eta^s\right)\left(\frac{M_t^s}{M_0^s}\right)\left[\frac{e^s\left(X_0\right)}{e^s\left(X_t\right)}\right]$$

Observations

- η^s typically negative (discounting)
- M^s is nongenderate when there are permanent shocks to $\log S$
- M^s induces a change of probability measure

See: Alvarez-Jermann (2005, Econometrica) and Hansen (2012, Econometrica) for more motivation

Related literature

- Ross (JF, 2015) explores asset pricing implications without the martingale component - recover probabilities from Arrow prices
- Borovicka, Hansen and Scheinkman (JF) show that the probability measure associated with M^s absorbs the risk adjustment for pricing growth-rate risk over (arbitrarily) long investment horizons
- Kazemi (RFS, 1992) and Alvarez and Jermann (2005, Econometrica) show that the one-period holding period return on a limiting long-term discount bond is

$$\exp(-\eta^s)\left[\frac{e^s(X_{t+1})}{e^s(X_t)}\right]$$

Observation: Last point opens the door to empirical estimation, testing and measuring - see Bakshi, Gurdip and Fousseni Chabi-Yo (JFE, 2012 and RFS, 2018) along with other references

Long-term risk return tradeoff

- ▷ Form:
$$\frac{S_t}{S_0} = \exp\left(t\eta^s\right)\left(\frac{M_t^s}{M_0^s}\right)\left[\frac{e^s\left(X_0\right)}{e^s\left(X_t\right)}\right]$$
- ▷ Form:
$$\frac{G_t}{G_0} = \exp\left(t\eta^g\right)\left(\frac{M_t^g}{M_0^g}\right)\left[\frac{e^g\left(X_0\right)}{e^g\left(X_t\right)}\right]$$
- ▷ Form:
$$\frac{S_t G_t}{S_0 G_0} = \exp\left(t\eta^{sg}\right)\left(\frac{M_t^{sg}}{M_0^{sg}}\right)\left[\frac{e^{sg}\left(X_0\right)}{e^{sg}\left(X_t\right)}\right]$$

Then
$$\eta^g + \eta^s - \eta^{sg}$$
is the limiting risk premium.

One-period asset pricing

Underlying setup

Suppose that

$$\begin{aligned} \log G_1 - \log G_0 &= \beta_g(X_0) + \alpha_g(X_0) \cdot W_1 \\ \log S_1 - \log S_0 &= \beta_s(X_0) + \alpha_s(X_0) \cdot W_1 \end{aligned}$$

The one-period return on this investment is the payoff in period one divided by the period-zero price:

$$R_1 \stackrel{\text{def}}{=} \frac{\left(\frac{G_1}{G_0}\right)}{\mathbb{E}\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) \mid \mathfrak{A}_0\right]}$$

One-period risk premium

Recall

$$R_1 \stackrel{\text{def}}{=} \frac{\left(\frac{G_1}{G_0}\right)}{\mathbb{E}\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) \mid \mathfrak{A}_0\right]}$$

Proportional risk premium

$$\begin{aligned}
&\log \mathbb{E}\left(R_1 \mid \mathfrak{A}_0\right) + \log \mathbb{E}\left(\frac{S_1}{S_0} \mid \mathfrak{A}_0\right) \\
&= \log \mathbb{E}\left(\frac{G_1}{G_0} \mid \mathfrak{A}_0\right) - \log \mathbb{E}\left(\frac{G_1 S_1}{G_0 S_0} \mid \mathfrak{A}_0\right) + \log \mathbb{E}\left(\frac{S_1}{S_0} \mid \mathfrak{A}_0\right) \\
&= -\alpha_s(X_0) \cdot \alpha_g(X_0)
\end{aligned}$$

$-\alpha_s(X_0)$ is the vector of risk prices.

Local perturbations

Parameterize a family of random variables $H_1(\mathsf{r})$ indexed by r using

$$\log H_1(\mathsf{r}) = \mathsf{r}\nu(X_0) \cdot W_1 - \frac{\mathsf{r}^2}{2} |\nu(X_0)|^2$$

where r is a scalar parameter. Normalize $\nu(X_0)$ so that

$$\mathbb{E}\left[|\nu(X_0)|^2\right] = 1.$$

With this parameterization,

$$\mathbb{E}\left[H_1(\mathsf{r}) | \mathfrak{A}_0\right] = 1.$$

Observation: Even when shocks are not normally distributed, we shall find it convenient to restrict $H_1(\mathsf{r})$ in this manner.

Local one-period asset prices

Given baseline payoff process G, form a family of payoffs $GH_1(\mathsf{r})$ with logarithmic increment:

$$\log G_1 - \log G_0 + \log H_1(\mathsf{r}) = \underbrace{\left[\alpha_g(X_0) + \mathsf{r}\nu(X_0)\right]}_{\text{new shock exposure}} \cdot W_1 + \beta_g(X_0) - \frac{\mathsf{r}^2}{2}\left|\nu(X_0)\right|^2 .$$

Proportional risk premium:

$$-\alpha_s(X_0)\left[\alpha_g(X_0) + \mathsf{r}\nu(X_0)\right].$$

The derivative with respect to r, $-\alpha_s(X_0) \cdot \nu(X_0)$, is the local price of exposure direction $\nu(X_0)$.

An alternative derivation

- shock exposure elasticity

$$\frac{d}{d\mathsf{r}} \log \mathbb{E}\left(\frac{G_1 H_1(\mathsf{r})}{G_0} \mid \mathfrak{A}_0 \right)\Bigg|_{\mathsf{r}=0} = \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{G_1}{G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{G_1}{G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

- shock cost elasticity

$$\frac{d}{d\mathsf{r}} \log \mathbb{E}\left(\frac{S_1 G_1 H_1(\mathsf{r})}{S_0 G_0} \mid \mathfrak{A}_0 \right)\Bigg|_{\mathsf{r}=0} = \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{S_1 G_1}{S_0 G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{S_1 G_1}{S_0 G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

- shock price elasticity is the difference

$$\nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{G_1}{G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{G_1}{G_0} \right) \mid \mathfrak{A}_0 \right]} \right) - \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{S_1 G_1}{S_0 G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{S_1 G_1}{S_0 G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

Multi-period asset pricing

Multi-period elasticities

- shock exposure elasticity

$$\frac{d}{d\mathsf{r}} \log \mathbb{E}\left(\frac{G_\tau H_\tau(\mathsf{r})}{G_0} \mid \mathfrak{A}_0 \right)\Bigg|_{\mathsf{r}=0} = \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{G_\tau}{G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{G_\tau}{G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

- shock cost elasticity

$$\frac{d}{d\mathsf{r}} \log \mathbb{E}\left(\frac{S_\tau G_\tau H_1(\mathsf{r})}{S_0 G_0} \mid \mathfrak{A}_0 \right)\Bigg|_{\mathsf{r}=0} = \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{S_\tau G_\tau}{S_0 G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{S_\tau G_\tau}{S_0 G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

- shock price elasticity is the difference

$$\nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{G_\tau}{G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{G_\tau}{G_0} \right) \mid \mathfrak{A}_0 \right]} \right) - \nu(X_0) \cdot \left(\frac{\mathbb{E}\left[\left(\frac{S_\tau G_\tau}{S_0 G_0} \right) W_1 \mid \mathfrak{A}_0 \right)}{\mathbb{E}\left[\left(\frac{S_\tau G_\tau}{S_0 G_0} \right) \mid \mathfrak{A}_0 \right]} \right)$$

Linear-quadratic specification

State evolution:

$$
\begin{aligned}
X_t^0 &= \bar{x} \\
X_{t+1}^1 &= \Theta_{10}^x + \Theta_{11}^x X_t^1 + \Sigma_{10}^x W_{t+1} \\
X_{t+1}^2 &= \Theta_{20}^x + \Theta_{21}^x X_t^1 + \Theta_{22}^x X_t^2 + \Theta_{23}^x \left(X_t^1 \otimes X_t^1 \right) \\
&\quad + \Sigma_{20}^x W_{t+1} + \Sigma_{21}^x \left(X_t^1 \otimes W_{t+1} \right) + \Sigma_{22}^x \left(W_{t+1} \otimes W_{t+1} \right)
\end{aligned}
$$

with Markov state $\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix}$.

Additive functional:

$$
\begin{aligned}
Y_{t+1} - Y_t = \Theta_0^y + \Theta_1^y X_{1,t} + \Theta_2^y X_{2,t} + \left(X_{1,t} \right)' \Theta_3^y X_{1,t} \\
+ \Sigma_0^y W_{t+1} + \left(X_{1,t} \right)' \Sigma_1^y W_{t+1} + \left(W_{t+1} \right)' \Sigma_2^y W_{t+1}
\end{aligned}
$$

A tractable computation

Suppose that

$$\log f(x) = \phi_o + \phi_1 \cdot x + \frac{1}{2} x^{1\prime} \Phi x^1$$

where $x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$. Then

$$\mathbb{E}\left[\exp\left(Y_{t+1} - Y_t\right) f(X_{t+1}) \mid X_t = x\right] = \hat{f}(x)$$

where

$$\log \hat{f}(x) = \hat{\phi}_o + \hat{\phi}_1 \cdot x + \frac{1}{2} x^{1\prime} \widehat{\Phi} x^1$$

for some scalar $\hat{\phi}_0$, vector $\hat{\phi}_1$, and matrix $\widehat{\Phi}$.

Define a valuation operator:

$$\mathbb{Q} f(x) \stackrel{\text{def}}{=} \mathbb{E}\left[\exp\left(Y_{t+1} - Y_t\right) f(X_{t+1}) \mid X_t = x\right]$$

Multi-period construction

From the Law of Iterated Expectations:

$$\begin{aligned}
\mathbb{Q}f(x) &= \mathbb{E}\left[\exp\left(Y_2 - Y_1\right) f(X_2) \mid X_1 = x\right] \\
\mathbb{Q}^2 f(x) &= \mathbb{E}\left[\exp\left(Y_3 - Y_1\right) f(X_2) \mid X_1 = x\right] \\
\mathbb{Q}^{\tau-1} f(x) &= \mathbb{E}\left[\exp\left(Y_\tau - Y_1\right) f(X_2) \mid X_1 = x\right]
\end{aligned}$$

Computing shock elasticities I

$$\frac{\mathbb{E}\left[\exp\left(Y_\tau - Y_0\right) W_1 \mid X_0\right)}{\mathbb{E}\left[\exp\left(Y_\tau - Y_0\right) \mid X_0 = x\right]}$$
$$= \mathbb{E}\left[\frac{\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1)W_1}{\mathbb{E}\left[\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1) \mid X_0 = x\right]} \mid X_0 = x\right]$$

Note that

$$\frac{\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1)}{\mathbb{E}\left[\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1) \mid X_0 = x\right]}$$

is positive and has conditional expectation equal to one.

Computing shock elasticities II

The random variable

$$\frac{\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1)}{\mathbb{E}\left[\exp(Y_1 - Y_0)\mathbb{Q}^{\tau-1}1(X_1) \mid X_0 = x\right]}$$

- is positive and conditional expectation equal to one
- is the exponential of a linear-quadratic function of the shock W_1
- induces a change of probability measure altering the conditional mean and covariance matrix of W_1 with a conditional mean that is affine in the state X_t^1.

Recursive utility example

Investor preferences represented as a homogeneous of degree one representation of the recursive utility continuation value is

$$V_t = \left[(1-\beta)\left(C_t\right)^{1-\rho} + \beta \left(R_t\right)^{1-\rho} \right]^{\frac{1}{1-\rho}}$$

where

$$R_t = \left(\mathbb{E}\left[\left(V_{t+1}\right)^{1-\gamma} \mid \mathfrak{A}_t \right] \right)^{\frac{1}{1-\gamma}} .$$

and

$0 < \beta < 1$ is the subjective discount factor

$\rho > 0$ and $\frac{1}{\rho}$ is the intertemporal elasticity of substitution

γ adjusts for "risk."

Use Bansal-Yaron calibration of the macro dynamics.

SDF process

The one-period increment in the stochastic discount factor process for recursive utility is:

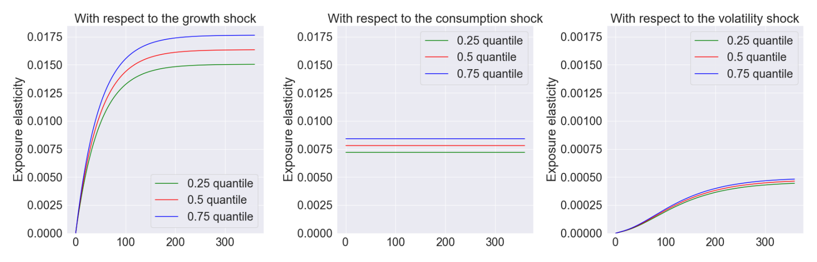
$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma} \left(\frac{V_{t+1}}{R_t}\right)^{\rho-1}$$

where

$$N^*_{t+1} = \left(\frac{V_{t+1}}{R_t}\right)^{1-\gamma}$$

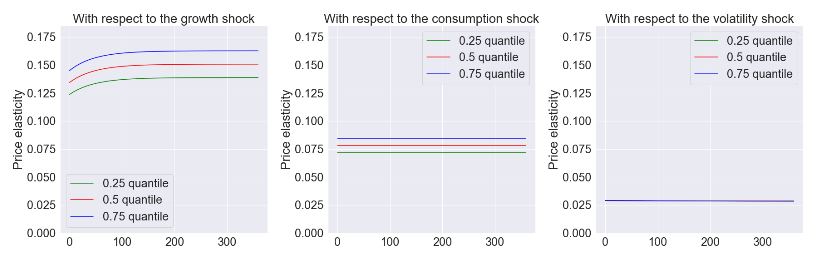
has conditional expectation equal to one.

Shock exposure elasticities



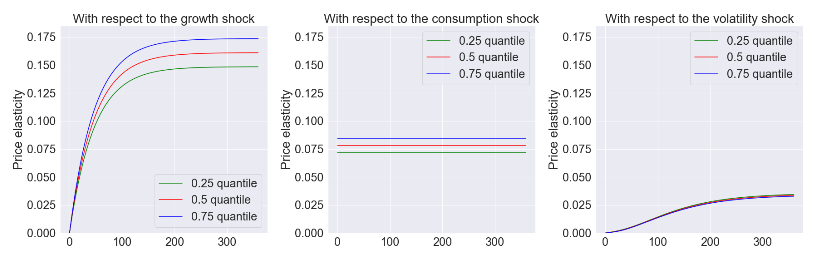
Time units are months.

Shock price elasticity $\rho = 1$



Time units are months. $\gamma = 10$, $\beta = .998$.

Shock price elasticity $\rho = 10$



Time units are months. $\gamma = 10$, $\beta = .998$.