CHAPTER 23

Risk Pricing over Alternative Investment Horizons*

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1. INTRODUCTION

Model-based asset prices are represented conveniently using stochastic discount factors. These discount factors are stochastic in order that they simultaneously discount the future and adjust for risk. Hansen and Richard (1987), Hansen and Jagannathan (1991), Cochrane (2001), and Singleton (2006) show how to construct and use stochastic discount factors to compare implications of alternative asset pricing models.

This chapter explores three interrelated topics using stochastic discount factors. First, I explore the impact of compounding stochastic discount factors over alternative investment horizons required for pricing asset payoff over multi-period investment horizons. The impact of compounding with state dependent discounting is challenging to characterize outside the realm of log-normal models. I discuss methods that push beyond log linear approximations to understand better valuation differences across models over alternative investment horizons. They allow for nonlinearities in the underlying stochastic evolution of the economy. As an important component of my discussion, I show how to use explicit models of valuation to extract the implications that are durable over long horizons by deconstructing stochastic discount factors in revealing ways.

State dependence in the growth of cash flows provides a second source of compounding. Thus I explore ways to characterize the pricing of growth rate risk by featuring the interaction between state dependence in discounting and growth. To support this aim I revisit the study of holding-period returns to cash flows over alternative investment horizons, and I suggest a characterization of the "term structure of risk prices" embedded in the valuation of cash flows with uncertain growth prospects. I obtain this second characterization by constructing elasticities that show how expected returns over different investment horizons respond to changes in risk exposures. Risk premia reflect both the exposure to risk and the price of that exposure. I suggest ways to quantify both of these channels of influence. In particular, I extend the concept of risk prices used to represent risk-return tradeoffs to study multi-period pricing and give a more complete understanding of alternative structural models of asset prices. By pricing the exposures of the shocks to the underlying macroeconomy, I provide valuation counterparts to impulse response functions used extensively in empirical macroeconomics.

In addition to presenting these tools, I also explore ways to compare explicit economic models of valuation. I consider models with varied specifications of investor preferences and beliefs, including models with habit persistent preferences, recursive utility preferences for which the intertemporal composition of risk matters, preferences that capture ambiguity aversion and concerns for model misspecification. I also explore how the dynamics of cross-sectional distribution of consumption influence valuation when complete risk sharing through asset markets is not possible. I consider market structures that acknowledge private information among investors or allow for limited commitment. I also consider structures that allow for solvency constraints and the preclusion of financial market contracting over idiosyncratic shocks.

The remainder of this chapter is organized as follows. In Section 2, I suggest some valuable characterizations of stochastic discount factor dynamics. I accomplish this in part by building a change of measure based on long-term valuation considerations in contrast to the familiar local risk-neutral change of measure. In Section 3, I extend the analysis by introducing a stochastic growth functional into the analysis. This allows for the interaction between stochastic components to discounting and growth over alternative investment or payoff horizons. I illustrate the resulting dynamic value decomposition (DVD) methods using some illustrative economies that feature the impact of investor preferences on asset pricing. Finally in Section 4, I consider some benchmark models with frictions to assess which frictions have only short-term consequences for valuation.

2. STOCHASTIC DISCOUNT FACTOR DYNAMICS

In this section we pose a tractable specification for stochastic discount factor dynamics that includes many of the parametric specifications in the literature. I then describe methods that characterize the implied long-term contributions to valuation and explore methods that help us characterize impact of compounding stochastic discount factors over multiple investment horizons.

2.1 Basic Setup

I begin with an information set $\mathcal{F}_0$ (sigma algebra) and two random vectors: Y_0 and X_0 that are $\mathcal{F}_0$ measurable. I consider an underlying stochastic process $(Y, X) = \{(Y_t, X_t) : t = 0, 1, \ldots\}$ and use this process to define an increasing sequence of information sets (a filtration) $\{\mathcal{F}_t : t = 0, 1, \ldots\}$ where (Y_u, X_u) is measurable with respect to $\mathcal{F}_t$ for $0 \leqslant u \leqslant t$. Following Hansen and Scheinkman (2012), I assume a recursive structure to the underlying stochastic process:

Assumption 2.1

The conditional distribution $(Y_{t+1} - Y_t, X_{t+1})$ conditioned on $\mathcal{F}_t$ depends only on X_t and is time invariant.

It follows from this assumption that *Y* does not "Granger cause" *X*, that *X* is itself a Markov process and that $\{Y_{t+1} - Y_t\}$ is a sequence of independent and identically distributed random vectors conditioned on the entire *X* process.[1]

I suppose that the processes that we use in representing asset values have a recursive structure.

[1] For instance, see Bickel, Ritov, and Ryden (1998). I may think of this conditional independence as being a more restrictive counterpart to Sims's (1972) alternative characterization of Granger (1969) causality.

Definition 2.2

An ***additive functional*** is a process whose first-difference has the form:

$$A_{t+1} - A_t = \kappa(Y_{t+1} - Y_t, X_{t+1}).$$

It will often be convenient to initialize the additive functional: $A_0 = 0$, but we allow for other initial conditions as well. I model stochastic growth and discounting using additive functionals after taking logarithms. This specification is flexible enough to include many commonly-used time series models. I relate the first-difference of A to the first-difference of Y in order to allow the increment in A to depend on the increment in Y in continuous-time counterparts.

2.2 A Convenient Factorization

Let S_t denote the stochastic discount factor between dates zero and t. The implicit discounting over a single time period between t and $t+1$ is embedded in this specification and is given by ratio $\frac{s_{t+1}}{s_t}$. The discounting is stochastic to accommodate risk adjustments in valuation. In representative consumer models with power utility functions

$$\frac{s_{t+1}}{s_t} = \exp(-\delta)\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}, \tag{1}$$

where C_t is aggregate consumption at date t, σ is the subjective rate of discount, and $\frac{1}{\rho}$ is the elasticity of intertemporal substitution. The formula on the right-hand side of (1) is the one-period intertemporal marginal rate of substitution for the representative consumer. This particular formulation is very special and problematic from an empirical perspective, but I will still use it as revealing benchmark for comparison.

One-period stochastic discount factors have been used extensively to characterize the empirical support, or lack thereof, for understanding one-period risk-return trade offs. My aim, however, is to explore valuation for alternative investment horizons. For instance, to study the valuation of date $t+\ 2$ payoffs from the vantage point of date t, I am led to compound 2 one-period stochastic discount factors:

$$\left(\frac{S_{t+2}}{S_{t+1}}\right)\left(\frac{S_{t+1}}{S_t}\right) = \frac{S_{t+2}}{S_t}.$$

Extending this logic leads me to the study of the stochastic discount factor process S, which embeds the stochastic discounting for the full array of investment horizons.

Alvarez and Jermann (2005), Hansen, Heaton, and Li (2008), Hansen and Scheinkman (2009), and Hansen (2012) suggest, motivate and formally defend a factorization of the form:

$$\frac{S_{t+2}}{S_t} = \exp(-\eta)\left(\frac{M_{t+1}}{M_t}\right)\left[\frac{f(X_{t+1})}{f(X_t)}\right], \tag{2}$$

where M is a martingale and X is a Markov process. I will show subsequently how to construct f. I will give myself flexibility in how I normalize S_0. While sometimes I will set it to one, any strictly positive normalization will suffice. In what follows we suppose that both log S and log M are additive functionals. Extending this formula to multiple investment horizons:

$$\frac{S_t}{S_0} = \exp(-\eta t) \left(\frac{M_t}{M_0} \right) \left[\frac{f(X_t)}{f(X_0)} \right]. \tag{3}$$

There are three components to this factorization, terms that I will interpret after I supply some more structure. Notice that each of the logarithms of each of the three components are themselves additive functionals.

I construct factorization (2) by solving the Perron–Frobenius problem:

$$E\left[\left(\frac{S_{t+1}}{S_t} \right) e(X_{t+1}) | X_t = x \right] = \exp(-\eta) e(x), \tag{4}$$

where e is a positive function of the Markov state. Then

$$\frac{M_t}{M_0} = \exp(\eta t) \left(\frac{S_t}{S_0} \right) \left[\frac{e(X_t)}{e(X_0)} \right]$$

is a martingale. Inverting this relation: gives (3) with $f = \frac{1}{e}$.

The preceding construction is not guaranteed to be unique. See Hansen and Scheinkman (2009) and Hansen (2012) for discussions. Recall that positive martingales with unit expectations can be used to induce alternative probability measures via a formula

$$E(M_t \psi_t | \mathcal{F}_0) = \widetilde{E}(\psi_t | \mathcal{F}_t)$$

for any bounded ψ_t that is in the date t information set (is $\mathcal{F}_t$ measurable). It is straightforward to show that under this change-of-measure, the process X remains Markov and that Assumption 2.1 continues to hold. This martingale construction is not guaranteed to be unique, however. There is at most one such construction for which the martingale M induces stochastically stable dynamics where stochastic stability requires:

Assumption 2.3

Under the change of probability measure,

$$\lim_{t \to \infty} \widetilde{E}\left[\phi(Y_t - Y_{t-1}, X_t) | X_0 = x \right] = \widetilde{E}\left[\phi(Y_t - Y_{t-1}, X_t) \right]$$

for any bounded Borel measurable function φ. The expectation on the right-hand side uses a stationary distribution implied by the change in the transition distribution.[2]

[2]One way to characterize the stationary distribution is to solve $E\left[\psi(X_0) M_0\right] = E\left(\widetilde{E}\left[\psi(X_1) | X_0 = x\right] M_0 \right)$.

See Hansen and Scheinkman (2009) and Hansen (2012) for discussions. There is a well developed set of tools for analyzing Markov processes that can be leveraged to check this restriction. See Meyn and Tweedie (1993) for an extensive discussion of these methods.

The version of factorization (2) that preserves this stochastic stability is of interest for the following reason. It allows me to compute:

$$E\left[S_t\phi\left(Y_t - Y_{t-1}, X_t\right)|X_0 = x\right] = \exp(-\eta t)e(x)\widetilde{E}\left[\frac{\phi(Y_t - Y_{t-1}, X_t)}{e(X_t)}\middle| X_0 = x\right].$$

Under stochastic stability,

$$\lim_{t\to\infty} \frac{1}{t}\log E\left[S_t\phi\left(Y_t - Y_{t-1}, X_t\right)|X_0 = x\right] = -\eta,$$

$$\lim_{t\to\infty} \log E\left[S_t\phi\left(Y_t - Y_{t-1}, X_t\right)|X_0 = x\right] + \eta t = \log e(x) + \log \widetilde{E}\left[\frac{\phi(Y_t - Y_{t-1}, X_t)}{e(X_t)}\right] \tag{5}$$

provided that $\phi > 0$. Thus the change-in-probability absorbs the martingale component to stochastic discount factors. The rate η is the long-term interest rate, which is evident from (5) when we set ϕ to be a function that is identically one.[3]

2.3 Other Familiar Changes in Measure

In the pricing of derivative claims, researchers often find it convenient to use the so-called "risk neutral" measure. To construct this in discrete time, form

$$\frac{M_{t+1}}{M_t} = \frac{S_{t+1}}{E(S_{t+1}|\mathcal{F}_t)}.$$

Then M is a martingale with expectation equal to one provided that $EM_0 = 1$. An alternative stochastic discount factor is:

$$\frac{S_{t+1}}{S_t} = \left(\frac{M_{t+1}}{M_t}\right) E\left(\frac{S_{t+1}}{S_t}\middle| \mathcal{F}_t\right). \tag{6}$$

The risk-neutral probability is the probability measure associated with the martingale M, and the one-period interest rate on a discount bond is:

$$-\log E\left(\frac{S_{t+1}}{S_t}\middle| \mathcal{F}_t\right).$$

[3] Following Backus et al. (1989), I have added sufficient structure as to provide a degenerate version of the Dybvig, Ingersoll, and Ross (1996) characterization of long-term rates. Dybvig et al. (1996) argue that long-term rates should be weakly increasing.

Absorbing the martingale into the change of measure, the one period prices are computed by discounting using the riskless rate, justifying the term "risk-neutral measure". Whenever the one-period interest rate is state independent, it is equal to η; and factorizations (2) and (6) coincide with $e = f = 1$ (or some other positive constant).

When interest rates are expected to vary over time, this variation in effect gives an adjustment for risk over multiple investment horizons. An alternative approch would be to use a different change of measure for each investment horizon, but this is not very convenient conceptually.[4] Instead I find it preferable to use a single change of measure with a constant adjustment to the long-term decay rate η in the stochastic discount factor that is state independent as in (2).

2.4 Log-Linear Models

It is commonplace to extract permanent shocks as increments in martingale components of time series. This approach is related to, but distinct from, the approach that I have sketched. The connection is closest when the underlying model of a stochastic discount factor is log-linear with normal shocks. See Alvarez and Jermann (2005) and Hansen et al. (2008). Suppose that

$$\log S_{t+1} - \log S_t = -\mu + H \cdot X_t + G \cdot W_{t+1},$$
$$X_{t+1} = AX_t + BW_{t+1},$$

where W is a multivariate sequence of standard normally distributed random vectors with mean zero and covariance I, and A is a matrix with stable eigenvalues (eigenvalues with absolute values that are strictly less than one). In this case we can construct a martingale component m in logarithms and

$$\log S_t - \log S_0 = -\nu t + m_t - m_0 + f \cdot X_t - h \cdot X_0,$$

where m is a an additive martingale satisfying:

$$m_{t+1} - m_t = \left[G' + H'(I - A)^{-1}B\right] W_{t+1},$$

and

$$f \cdot X_t = -H'(I - A)^{-1}X_t.$$

Increments to the additive martingale are permanent shocks, and shocks that are uncorrelated have only transient consequences.

[4] Such changes in measure are sometimes called forward measures. See Jamshidian (1989) for an initial application of these measures.

While m is an additive martingale, $\exp(m)$ is not a martingale. It is straightforward to construct the martingale M by forming

$$\frac{M_t}{M_0} = \exp(m_t - m_0) \exp\left[-\frac{t}{2}|G' + H'(I - A)^{-1}B|^2\right],$$

where the second term adjusts is a familiar log-normal adjustment. With stochastic volatility models or regime-shift models, the construction is not as direct. See Hansen (2012) for a discussion of a more general link be between martingale constructions for additive processes and factorization (3).[5]

2.5 Model-Based Factorizations

Factorization (3) provides a way to formalize long-term contributions to valuation. Consider two alternative stochastic discount factor processes, S and S^*, associated with two different models of valuation.

Definition 2.4

The valuation implications between model S and S^* are ***transient*** if these processes share a common value of the long-term interest rate η and the martingale component M.

Consider the factorization (3) for the power utility model mentioned previously:

$$S_t^* = \exp(-\delta t)\left(\frac{C_t}{C_0}\right)^{-\rho} = \exp(-\eta^* t)\left(\frac{m_t^*}{m_0^*}\right)\left[\frac{f^*(X_t)}{f^*(X_0)}\right], \tag{7}$$

where δ is the subjective rate of discount, $\rho > 0$, and $\left(\frac{C_t}{C_0}\right)^{-\rho}$ is the (common) intertemporal marginal rate of substitution of an investor between dates zero and t. I assume that $\log C$ satisfies Assumption 2.1. It follows immediately the logarithm of the marginal utility process, $\gamma \log C$ satisfies this same restriction. In addition the function $f^* = \frac{1}{e^*}$ and (e^*, η^*) solves the eigenvalue equation (4) including the imposition of stochastic stability.

Suppose for the moment we hold the consumption process fixed as a device to understand the implications of changing preferences. Bansal and Lehmann (1997) noted that the stochastic discount factors for many asset pricing models have a common structure. I elaborate below. The one-period ratio of the stochastic discount factor is:

$$\frac{S_{t+1}}{S_t} = \left(\frac{S_{t+1}^*}{S_t^*}\right)\left[\frac{h(X_{t+1})}{h(X_t)}\right]. \tag{8}$$

[5] The martingale extraction in logarithms applies to a much larger class of processes and results in an additive functional. The exponential of the resulting martingale shares a martingale component in the level factorization (3) with the original process.

From this baseline factorization,

$$\frac{S_t}{S_0} = \exp(-\eta t) \left(\frac{M_t}{M_0} \right) \left[\frac{f(X_t)h(X_t)}{f(X_0)h(X_0)} \right].$$

The counterpart for the eigenfunction e is $\frac{1}{f^* h}$. Thus when factorization (8) is satisfied, the long-term interest rate η and the martingale component to the stochastic discount factor are the same as those with power utility. The function h contributes "transient" components to valuation. Of course these transient components could be highly persistent.

While my aim is to provide a fuller characterization of the impact of the payoff horizon on the compensation for exposure to risk, locating permanent components to models of valuation provides a good starting point. It is valuable to know when changes in modeling ingredients have long-term consequences for valuation and when these changes are more transient in nature. It is also valuable to understand when "transient changes" in valuation persist over long investment horizons even though the consequences eventually vanish. The classification using martingale components is merely an initial step for a more complete understanding.

I now explore the valuation implications of some alternative specifications of investor preferences.

2.5.1 Consumption Externalities and Habit Persistence

See Abel (1990), Campbell and Cochrane (1999), Menzly, Santos, and Veronesi (2004), and Garcia, Renault, and Semenov (2006) for representations of stochastic discount factors in the form (8) for models with history dependent measures of consumption externalities. A related class of models are those in which there are intertemporal complementaries in preferences of the type suggested by Sundaresan (1989), Constantinides (1990), and Heaton (1995). As argued by Hansen et al. (2008) these models also imply stochastic discount factors that can be expressed as in (8).

2.5.2 Recursive Utility

Consider a discrete-time specification of recursive preferences of the type suggested by Kreps and Porteus (1978) and Epstein and Zin (1989). I use the homogeneous-of-degree-one aggregator specified in terms of current period consumption C_t and the continuation value V_t for a prospective consumption plan from date t forward:

$$V_t = \left[(\varsigma C_t)^{1-\rho} + \exp(-\delta) \left[\mathcal{R}_t(V_{t+1}) \right]^{1-\rho} \right]^{\frac{1}{1-\rho}}, \tag{9}$$

where

$$\mathcal{R}_t(V_{t+1}) = \left(E\left[(V_{t+1})^{1-\gamma} | \mathcal{F}_t \right] \right)^{\frac{1}{1-\gamma}}$$

adjusts the continuation value V_{t+1} for risk. With these preferences, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution and δ is a subjective discount rate. The parameter ς does not alter preferences, but gives some additional flexibility, and we will select it in a judicious manner. The stochastic discount factor S for the recursive utility model satisfies:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left[\frac{V_{t+1}/C_{t+1}}{\mathcal{R}_t(V_{t+1}/C_t)}\right]^{\rho-\gamma}. \tag{10}$$

The presence of the next-period continuation value in the one-period stochastic discount factor introduces a forward-looking component to valuation. It gives a channel by which investor beliefs matter. I now explore the consequences of making the forward-looking contribution to the one-period stochastic discount factor as potent as possible in a way that can be formalized mathematically. This is relevant for the empirical literature as that literature is often led to select parameter configurations that feature the role of continuation values.

Following Hansen (2012) and Hansen and Scheinkman (2012), we consider the following equation:

$$E\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\hat{e}(X_{t+1})|X_t = x\right] = \exp(\hat{\eta})\hat{e}(x).$$

Notice that this eigenvalue equation has the same structure as (4) with $(C_t)^{1-\gamma}$ taking the place of S_t. The formula for the stochastic discount factor remains well defined in the limiting case as we let $(\varsigma)^{1-\rho}$ tend to zero and δ decreases to[6]

$$\frac{1-\rho}{1-\gamma-1}\hat{\eta}.$$

Then

$$\frac{V_t}{C_t} \approx \left[\hat{e}(X_t)\right]^{1-\gamma},$$

and

$$S_t \approx \exp(-\hat{\eta}t)\left(\frac{C_t}{C_0}\right)^{-\gamma}\left[\frac{\hat{e}(X_t)}{\hat{e}(X_0)}\right]^{\frac{\rho-\gamma}{1-\gamma}}. \tag{11}$$

[6] Hansen and Scheinkman (2012) use the associated change of measure to show when existence to the Perron–Frobenius problem implies the existence of a solution to the fixed point equation associated with an infinite-horizon investor provided that δ is less than this limiting threshold.

Therefore, in the limiting case

$$h(x) = \hat{e}(x)^{\frac{\rho-\gamma}{1-\gamma}}$$

in (8).

2.5.3 Altering Martingale Components

Some distorted belief models of asset pricing feature changes that alter the martingale components. As I have already discussed, positive martingales with unit expectations imply changes in the probability distribution. They act as so-called Radon–Nikodym derivatives for changes that are absolutely continuous over any finite time interval. Suppose that N is a martingale for which $\log\ N$ is an additive functional. Thus

$$E\left(\frac{N_{t+1}}{N_t}\middle| X_t = x\right) = 1.$$

This martingale captures investors beliefs that can be distinct from those given by the underlying model specification. Since Assumption 2.1 is satisfied, for the baseline specification, it may be shown that the alternative probability specification induced by the martingale N also satisfies the assumption. This hypothesized difference between the model and the beliefs of investors is presumed to be permanent with this specification. That is, investors have confidence in this alternative model and do not, for instance, consider a mixture specification while attempting to infer the relative weights using historical data.

For some distorted belief models, the baseline stochastic discount factor S^* from power utility is altered by the martingale used to model the belief distortion:

$$S = S^* N.$$

Asset valuation inherits the distortion in the beliefs of the investors. Consider factorization (7) for S^*. Typically NM^* will not be a martingale even though both components are martingales. Thus to obtain the counterpart factorization for a distorted belief economy with stochastic discount factor S requires that we extract a martingale component from NM^*. Belief changes of this type have permanent consequences for asset valuation.

Examples of models with exogenous belief distortions that can be modeled in this way include Cecchetti, Lam, and Mark (2000) and Abel (2002). Related research by Hansen, Sargent, and Tallarini (1999), Chen and Epstein (2002), Anderson, Hansen, and Sargent (2003), and Ilut and Schneider (2012) uses a preference for robustness to model misspecification and ambiguity aversion to motivate explicitly this pessimism.[7] In this

[7] There is a formal link between some recursive utility specifications and robust utility specifications that has origins in the control theory literature on risk-sensitive control. Anderson et al. (2003) and Maenhout (2004) develop these links in models of portfolio choice and asset pricing.

literature the form of the pessimism is an endogenous response to investors' uncertainty about which among a class of model probability specifications governs the dynamic evolution of the underlying state variables. The martingale N is not their "actual belief", but rather the outcome of exploring the utility consequences of considering an array of probability models. Typically there is a benchmark model that is used, and we take the model that we have specified without distortion as this benchmark. In these specifications, the model uncertainty does not vanish over time via learning because investors are perpetually reluctant to embrace a single probability model.

2.5.4 Endogenous Responses

So far our discussion has held fixed the consumption process in order to simplify the impact of changing preferences. Some stochastic growth models with production have a balanced growth path relative to some stochastically growing technology. In such economies, some changes in preferences, while altering consumption allocations, may still preserve the martingale component along with the long-term interest rate.

2.6 Entropy Characterization

In the construction that follows we build on ideas from Bansal and Lehmann (1997), Alvarez and Jermann (2005), and especially Backus, Chernov, and Zin (2011). The relative entropy of a stochastic discount factor functional S for horizon t is given by:

$$\frac{1}{t}\left[\log E(S_t|X_0 = x) - E\left(\log S_t|X_0 = x\right)\right],$$

which is nonnegative as an implication of Jensen's Inequality. When S_t is log-normal, this notion of entropy yields one-half the conditional variance of $\log S_t$ conditioned on date zero information, and Alvarez and Jermann (2005) propose using this measure as a "generalized notion of variation". Backus et al. (2011) study this measure of relative entropy averaged over the initial state X_0. They view this entropy measure for different investment horizons as an attractive alternative to the volatility of stochastic discount factors featured by Hansen and Jagannathan (1991). To relate these entropy measures to asset pricing models and data, Backus et al. (2011) note that

$$-\frac{1}{t}E\left[\log E(S_t|X_0)\right]$$

is the average yield on a t-period discount bond where we use the stationary distribution for X_0. Following Bansal and Lehmann (1997),

$$-\frac{1}{t}E(\log S_t) = -E(\log S_1),$$

is the average one-period return on the maximal growth portfolio under the same distribution.

Borovicka and Hansen (2012) derive a more refined quantification of how entropy depends on the investment horizon t given by

$$\frac{1}{t}\left[\log E(S_t|X_0) - E(\log S_t|X_0)\right] = \frac{1}{t}\sum_{j=1}^{t} E\left[\varsigma(X_{t-j}, j)|X_0\right]. \tag{12}$$

The right-hand side represents the horizon t entropy in terms of averages of the building blocks $\varsigma(x,t)$ where

$$\varsigma(x,t) = \log E\left[S_t|X_0 = x\right] - E\left[\log E(S_t|\mathcal{F}_1)|X_0 = x\right] \geqslant 0.$$

The term ς is itself a measure of "entropy" of

$$\frac{E(S_t|\mathcal{F}_1)}{E(S_t|\mathcal{F}_0)}$$

conditioned on date zero information and measures the magnitude of new information that arrives between date zero and date one for S_t. For log-normal models, $\varsigma(x,t)$ is one-half the variance of $E(\log S_t|\mathcal{F}_1) - E(\log S_t|\mathcal{F}_0)$.

3. CASH-FLOW PRICING

Rubinstein (1976) pushed us to think of asset pricing implications from a multi-period perspective in which an underlying set of future cash flows are priced. I adopt that vantage point here. Asset values can move, either because market-determined stochastic discount rates have altered (a price change), or because the underlying claim implies a higher or lower cash flow (a quantity change). These two channels motivate formal methods for enhancing our understanding of what economic models have to say about present-value relations. One common approach uses a log-linear approximation to identify two (correlated) sources of time variation in the ratio of an asset value to the current period cash flow. The first source is time variation in expected returns to holding the asset, a price effect, and the second is time variation in expected dividend growth rates, a quantity effect. Here I explore some more broadly applicable methods to produce "dynamic valuation decompositions" which are complementary to the log-linear approach. My aim is to unbundle the pricing of cash flows in revealing ways. The specific impetus for this formulation comes from the work of Lettau and Wachter (2007) and Hansen et al. (2008), and the general formulation follows Hansen and Scheinkman (2009), and Hansen (2012).

3.1 Incorporating Stochastic Growth in the Cash Flows

Let G be a stochastic growth factor where $\log G$ satisfies Assumption 2.1. Notice that if $\log G$ and $\log S$ both satisfy this assumption, their sum does as well. While the stochastic discount factor decays over time, the stochastic growth factor grows over time. I will presume that discounting dominates and that the product SG is expected to decay over time. I consider cash flows of the type:

$$G_{t+1}\phi(Y_{t+1} - Y_t, X_{t+1}), \tag{13}$$

where G_0 is in the date zero information set $\mathcal{F}$. The date t value of this cash flow is:

$$E\left[\frac{S_{t+1}}{S_0}G_{t+1}\phi(Y_{t+1}-Y_t, X_{t+1})|\mathcal{F}_0\right] = G_0 E\left[\frac{S_{t+1}\,G_{t+1}}{S_0\,G_0}G_{t+1}\phi(Y_{t+1}-Y_t, X_{t+1})|X_0\right].$$

An equity sums the values of the cash flows at all dates $t = 1, 2, \ldots$ By design we may compute values recursively by repeatedly applying a one-period valuation operator:

$$\mathbb{V}h(x) = E\left[\frac{S_{t+1}\,G_{t+1}}{S_t\,G_t}h(x_{t+1})|X_t = x\right].$$

Let

$$h(x) = E\left[\frac{S_{t+1}\,G_{t+1}}{S_t\,G_t}\phi(Y_{t+1} - Y_t, X_{t+1})|X_t = x\right].$$

Then

$$E\left[\frac{S_{t+1}}{S_0}G_{t+1}\phi(Y_{t+1} - Y_t, X_{t+1})|\mathcal{F}_0\right] = G_0\mathbb{V}^t h(x).$$

To study cash flow pricing with stochastic growth factors, we use a factorization of the type given in (3) but applied to SG instead of S:

$$\frac{S_t\,G_t}{S_o\,G_0} = \exp(-\eta t)\left(\frac{M_t}{M_0}\right)\left[\frac{f(X_t)}{f(X_0)}\right],$$

where $f = \frac{1}{e}$ and e solves:

$$E\left[\left(\frac{S_{t+1}\,G_{t+1}}{S_t\,G_t}\right)e(X_{t+1})|X_t = x\right] = \exp(-\eta)e(x).$$

The factorization of SG *cannot* be obtained by factoring S and G separately and multiplying the outcome because products of martingales are not typically martingales. Thus co-dependence matters.[8]

[8] When S and G are jointly lognormally distributed, we may first extract martingale components of log S and log G and add these together and exponentiate. While this exponential will not itself be a martingale, we may construct a positive martingale by multiplying this exponential by a geometrically declining scale factor.

3.2 Holding-Period Returns on Cash Flows

A return to equity with cash flows or dividends that have stochastic growth components can be viewed as a bundle of portfolios of holding-period returns on cash flows with alternative payout dates (see Hansen et al. 2008; Lettau and Wachter, 2007). The gross one-period holding-period return over a payoff horizon t is:

$$\left(\frac{G_1}{G_0}\right)\left(\frac{\mathbb{V}_{t-1}\left[h(X_1)\right]}{\mathbb{V}_t\left[h(X_0)\right]}\right).$$

Changing the payoff date t changes the exposure through a valuation channel as reflected by the second term in brackets, while the direct cash flow channel reflected by the first term remains the same as we change the payoff horizon.

To characterize the holding-period return for large t, I apply the change in measure and represent this return as:

$$\exp(\eta)\frac{G_1}{G_0}\left[\frac{e(X_1)}{e(X_0)}\right]\left(\frac{\widetilde{E}\left[h(X_t)f(X_t)|X_1\right]}{\widetilde{E}\left[h(X_t)f(X_t)|X_0\right]}\right).$$

The last term converges to unity as the payoff horizon τ increases, and the first two terms do not depend on τ. Thus the limiting return is:

$$\left(\frac{G_1}{G_0}\right)\left[\exp(\eta)\frac{e(X_1)}{e(X_0)}\right]. \tag{14}$$

The valuation component is now tied directly to the solution to the Perron–Frobenius problem. An eigenfunction ratio captures the state dependence. In addition there is an exponential adjustment η, which is in effect a value-based measure of duration of the cash flow G and is independent of the Markov state. When η is near zero, the cash-flow values deteriorate very slowly as the investment horizon is increased.

The study of holding-period returns on cash flows payoffs over alternative payoff dates gives one way to characterize a valuation dynamics. Recent work by van Binsbergen, Brandt, and Koijen (2012) develops and explores empirical counterpart to these returns. Next I appeal to ideas from price theory to give a different depiction.

3.3 Shock Elasticities

Next I develop valuation counterparts to impulse-response functions commonly used in the study of dynamic, stochastic equilibrium models. I refer to these counterparts as shock elasticities. As I will show, these elasticities measure both exposure and price sensitivity over alternative investment horizons.

As a starting point, consider a cash flow G and stochastic discount factor S. For investment horizon t, form the logarithm of the expected return to this cash flow given by:

$$\log E\left[\left(\frac{G_t}{G_0}\right)\Big| X_0 = x\right] - \log E\left[\left(\frac{G_t}{G_0}\right)\left(\frac{S_t}{S_0}\right)\Big| X_0 = x\right],$$

where the scaling by G_0 is done for convenience. The first term is the logarithm of the expected payoff and the second term is the logarithm of the price. To measure the risk premium I compare this expected return to a riskless investment over the same time horizon. This is a special case of my previous calculation in which I set $G_t = 1$ for all t. Thus the logarithm of this returns is:

$$-\log E\left[\left(\frac{S_t}{S_0}\right)\Big| X_0 = x\right].$$

I measure the risk premium by comparing these two investments:

$$\begin{aligned}\textbf{risk premium} = & \log E\left[\left(\frac{G_t}{G_0}\right)|X_0 = x\right] - \log E\left[\left(\frac{G_t}{G_0}\right)\left(\frac{S_t}{S_0}\right)|X_0 = x\right] \\ & + \log E\left[\left(\frac{S_t}{S_0}\right)|X_0 = x\right].\end{aligned} \tag{15}$$

The last two terms, taken together, denote the (logarithm of the) futures price of a dividend contract that pays a dividend at date t. (See van Binsbergen et al. (2012) for empirical measures of closely related futures price.) In what follows I will study the value implications as measured by what happens to the risk premium when I perturb the exposure of the cash flow to the underlying shocks.

To unbundle value implications, I borrow from price theory by computing shock price and shock exposure elasticities. (I think of an exposure elasticity as the counterpart to a quantity elasticity.) In so doing I build on the continuous-time analyses of Hansen and Scheinkman (2012a) and Borovićka, Hansen, Hendricks, and Scheinkman (2011) and on the discrete-time analysis of Borovićka and Hansen (2012). To simplify the interpretation, suppose there is an underlying sequence of iid multivariate standard normally distributed shocks $\{W_{t+1}\}$. Introduce:

$$\log H_{t+1}(r) - \log H_t(r) = r\sigma(X_t) \cdot W_{t+1} - \frac{(r)^2}{2}|\sigma(X_t)|^2,$$

where I assume that

$$E\left[|\sigma(X_t)|^2\right] = 1$$

and $\log H_0(r) = 0$. Here I use $\sigma(x)$ to select the combination of shocks that is of interest and I scale this state-dependent vector in order that $\sigma(X_t) \cdot W_{t+1}$ has a unit standard deviation.[9]

[9] Borovićka et al. (2011) suggest counterpart elasticities for discrete states modeled as Markov processes.

Also I have constructed the increment in $\log H_{t+1}$ so that

$$E\left[\frac{H_{t+1}(r)}{H_t(r)}\middle| X_t = x\right] = 1.$$

I use the resulting process $H(\mathsf{r})$ to define a scalar family of martingale perturbations parameterized by r.

Consider a cash flow G that may grow stochastically over time. By multiplying G by $H(r)$, I alter the exposure of the cash flow to shocks. Since I am featuring small changes, I am led to use the process:

$$D_{t+1} - D_t = \sigma(X_t) \cdot W_{t+1}$$

with $D_0 = 0$ to represent two exposure elasticities:

$$\epsilon_e(x,t) = \frac{\mathsf{d}}{\mathsf{dr}}\frac{1}{t}\log E\left[\frac{G_t}{G_0}H_t(\mathsf{r})|X_0 = x\right]\Bigg|_{\mathsf{r}=0} = \frac{1}{t}\frac{E\left[\frac{G_t}{G_0}D_t|X_0 = x\right]}{E\left[\frac{G_t}{G_0}|X_0 = x\right]},$$

$$\varepsilon_e(x,t) = \frac{\mathsf{d}}{\mathsf{dr}}\log E\left[\frac{G_t}{G_0}H_1(\mathsf{r})|X_0 = x\right]\Bigg|_{\mathsf{r}=0} = \sigma(X_0)\frac{E\left[\frac{G_t}{G_0}W_1|X_0 = x\right]}{E\left[\frac{G_t}{G_0}|X_0 = x\right]}.$$

These elasticities depend both on the investment horizon t and the current value of the Markov state x. For a fixed horizon t, the first of these elasticities, which I call a risk-price elasticity, changes the exposure at all horizons. The second one concentrates on changing only the first period exposure, much like an impulse response function.[10] As argued by Borovićka et al. (2011) and Borovicka and Hansen (2012), the risk price-elasticities are weighted averages of the shock-price elasticities.

The long-term limit (as $t \to \infty$) of the shock-price elasticity has a tractable characterization. Consider a factorization of the form (3), but applied to G. Using the martingale from this factorization, Borovicka and Hansen (2012) show that

$$\lim_{t\to\infty}\frac{E\left[\frac{G_t}{G_0}W_1|X_0 = x\right]}{E\left[\frac{G_t}{G_0}|X_0 = x\right]} = \widetilde{E}\left[W_1|X_0 = x\right].$$

[10] Under log-normality there is a formal equivalence between our elasticity and an impulse response function.

As intermediate calculations, I also compute:

$$\epsilon_v(x,t) = \frac{\mathrm{d}}{\mathrm{dr}}\frac{1}{t}\log E\left[\frac{S_t G_t}{S_0 G_0} H_t(\mathsf{r}) | X_0 = x\right]\Bigg|_{r=0} = \frac{1}{t}\frac{E\left[\frac{S_t G_t}{S_0 G_0} D_t | X_0\right]}{E\left[\frac{S_t G_t}{S_0 G_0} | X_0\right]},$$

$$\varepsilon_v(x,t) = \frac{\mathrm{d}}{\mathrm{dr}}\log E\left[\frac{S_t G_t}{S_0 G_0} H_1(\mathsf{r}) | X_0 = x\right]\Bigg|_{r=0} = \frac{E\left[\frac{S_t G_t}{S_0 G_0} D_1 | X_0\right]}{E\left[\frac{S_t G_t}{S_0 G_0} | X_0\right]},$$

which measure the sensitivity of value to changes in the exposure. These elasticities incorporate both a change in price and a change in exposure. The implied risk-price and shock-price elasticities are given by:

$$\epsilon_p(x,t) = \epsilon_e(x,t) - \epsilon_v(x,t),$$
$$\varepsilon_p(x,t) = \varepsilon_e(x,t) - \varepsilon_v(x,t).$$

In what follows I draw on some illustrations from the existing literature.

3.3.1 Lettau–Wachter Example

Lettau and Wachter (2007) consider an asset pricing model of cash-flow duration. They use an ad hoc model of a stochastic discount factor to display some interesting patterns of risk premia. When thinking about the term structure of risk premia, I find it useful to distinguish pricing implications from exposure implications. Both can contribute to risk premia as a function of the investment horizon.

Lettau and Wachter (2007) explore implications of a cash-flow process with linear dynamics:

$$X_{t+1} = \begin{bmatrix} .9658 & 0 \\ 0 & .9767 \end{bmatrix} X_t + \begin{bmatrix} .12 & 0 & 0 \\ 0 & -.0013 & .0009 \end{bmatrix} W_{t+1},$$

where $\{W_{t+1}\}$ is i.i.d. multivariate standard normally distributed. They model the logarithm of the cash flow process as

$$\log G_{t+1} - \log G_t = \mu_g + X_t^{[2]} + \begin{bmatrix} 0 & .0724 & 0 \end{bmatrix} W_{t+1},$$

where $X_t^{[2]}$ is the second component of X_t. I compute shock exposure elasticities, which in this case are essentially the same as impulse response functions for log G since the cash flow process is log-normal. The exposure elasticities for the two shocks are depicted in the top panel of Figure 1.

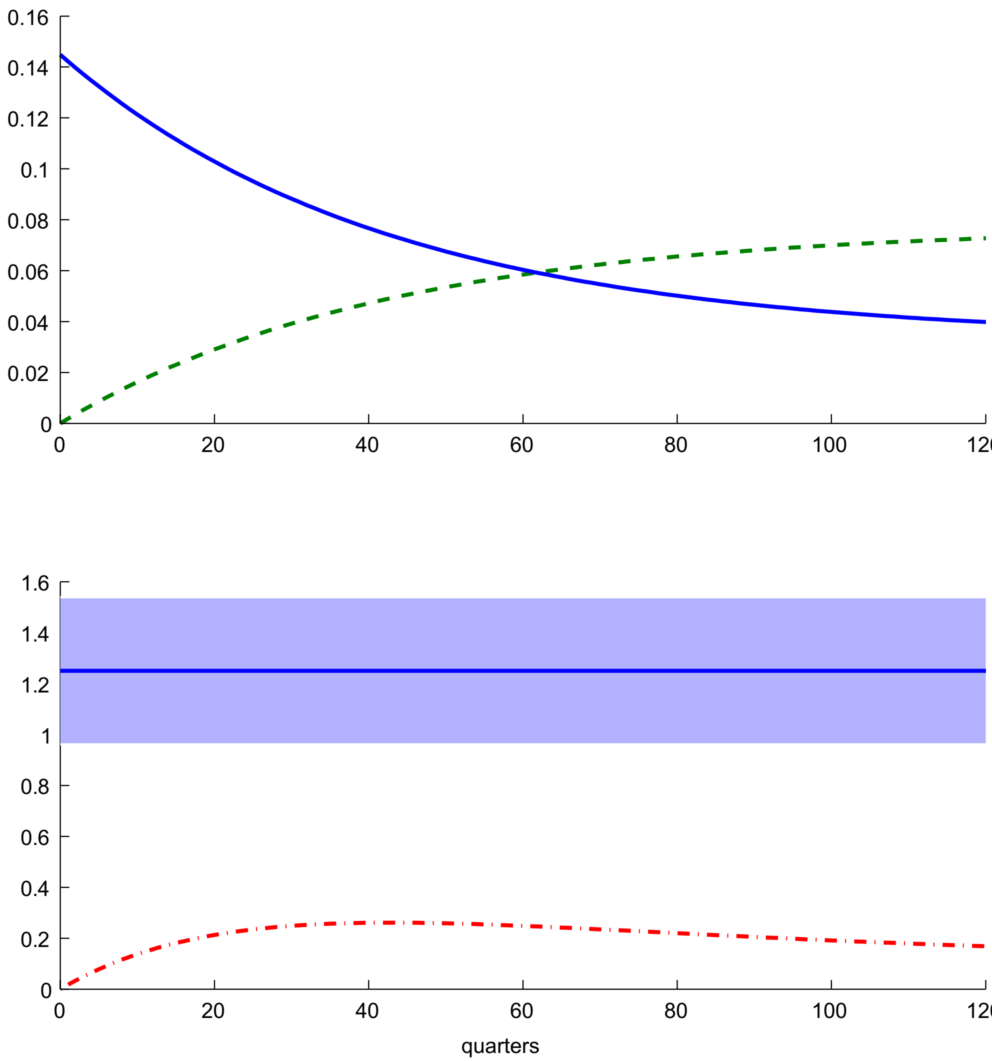

Figure 1 The top panel of this figure depicts the shock-exposure elasticities for the second (solid line) and third (dashed line) shocks obtained by setting σ to be the corresponding coordinate vectors. The shock-exposure elasticities for the first shock are zero. The bottom panel of this figure depicts the shock-price elasticities for the first shock (dotted line) and for the second shock (solid line) over alternative investment horizons. The shock-price elasticities for the third shock are zero. The shaded area gives the interquartile range for the shock-price elasticities implied by state dependence.

For shock two, the immediate exposure dominates that long-run response. In contrast the third shock exposure starts at zero and builds to a positive limit, but at a value that is notably higher than the second shock.

Next we assign "prices" to the shock exposures. The stochastic discount factor in Lettau–Wachter model evolves as:

$$\log S_{t+1} - \log S_t = -r - \left(.625 + X_t^{[1]}\right)\begin{bmatrix}0 & 1 & 0\end{bmatrix} W_{t+1} - \frac{\left|.625 + X_t^{[1]}\right|^2}{2}.$$

Nonlinearity is present in this model because the conditional mean of $\log S_{t+1} - \log S_t$ is quadratic in $X_t^{[1]}$. This is a model with a constant interest rate r and state dependent one-period shock price-vector:

$$\left(.625 + X_t^{[1]}\right)\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

By assumption, only the second shock commands a nonzero one-period shock price elasticity and this elasticity varies over time. The process $\{.625 + X_t^{[1]}\}$ is a stochastic volatility process that induces movements in the shock price elasticities. In its stationary distribution, this process has a standard deviation of .46 and hence varies substantially relative to its mean of .625. The first shock alters the first component of X_t and the shock-price elasticity for the first shock is different from zero after one period. The cash flow G does not respond to this shock so the "pricing" of the first component of $W_{t+1}^{[1]}$ does not play a direct role in the valuation of G.[11]

The shock-price elasticities are depicted in the bottom panel of Fig 1. A consequence of the specification of the stochastic discount factor S is that the second shock has a constant (but state dependent) shock-price elasticity of $.625 + X_t^{[1]}$ as a function of the investment horizon. This shock has the biggest impact for the cash flow, and it commands the largest shock price elasticity, both immediately and over the long term. Thus, I have shown that this application of dynamic value decomposition reveals that the impetus for the downward risk premia as a function of horizon comes from the dynamics of the cash-flow shock exposure and not from the price elasticity of that exposure.

We now shift to a different specification of preferences and cash flows, and show what the same methods reveal in a different context.

3.3.2 Recursive Utility

We illustrate pricing implications for the recursive utility model using a specification from Hansen, et al. (2007) of a "long-run risk" model for consumption dynamics featured by Bansal and Yaron (2004). Bansal and Yaron (2004) use historical data from the

[11] Lettau and Wachter (2007) use this model to interpret the differential expected returns in growth and value stocks. Value stocks are more exposed to the second shock.

United States to motivate their model including the choice of parameters. Their model includes predictability in both conditional means and in conditional volatility. We use the continuous-time specification from Hansen et al. (2007) because the continuous-time specification of stochastic volatility is more tractable:

$$
\begin{aligned}
\mathrm{d}X_t^{[1]} &= -.021 X_t^{[1]}\,\mathrm{d}t + \sqrt{X_t^{[2]}}\begin{bmatrix}.00031 & -.00015 & 0\end{bmatrix}\mathrm{d}W_t,\\
\mathrm{d}X_t^{[2]} &= -.013\left(X_t^{[2]} - 1\right)\mathrm{d}t + \sqrt{X_t^{[2]}}\begin{bmatrix}0 & 0 & -.038\end{bmatrix}\mathrm{d}W_t,\\
\mathrm{d}\log C_t &= .0015\,\mathrm{d}t + X_t^{[1]}\,\mathrm{d}t + \sqrt{X_t^{[2]}}\begin{bmatrix}.0034 & 0.007 & 0\end{bmatrix}\mathrm{d}W_t,
\end{aligned}
$$

where W is a trivariate standard Brownian motion. The unit of time in this time series specification is 1 month, although for comparability with other models I plot shock-price elasticities using quarters as the unit of time. The first component of the state vector is the state dependent component to the conditional growth rate, and the second component is a volatility state. Both the growth state and the volatility state are persistent. We follow Hansen (2012) in configuring the shocks for this example. The first one is the "permanent shock" identified using standard time series methods and normalized to have a unit standard deviation. The second shock is a so-called temporary shock, which by construction is uncorrelated with the first shock.

Our analysis assumes a discrete-time model. A continuous-time Markov process X observed at say interval points in time remains a Markov process in discrete time. Since $\log C_{t+1} - \log C_t$ is constructed via integration, it is not an exact function of X_{t+1} and X_t. To apply our analysis, we define $Y_{t+1} = \log C_{t+1} - \log C_t$. Given the continuous-time Markov specification, the joint distribution of $\log C_{t+1} - \log C_t$ and X_{t+1} conditioned on past information only depends on the current Markov state X_t as required by Assumption 2.1.[12] The resulting shock-price elasticities are reported in Figure 2 for the three different shocks. Since the model with power utility ($\rho = \gamma = 8$) has preferences that are additively separable, the pricing impact of a permanent shock or a stochastic-volatility shock accumulates over time with the largest shock-price elasticities at the large investment horizon limit. In contrast, recursive utility with ($\rho = 1, \gamma = 8$) has an important forward-looking component for pricing.[13] As a consequence, the trajectory for the shock-price elasticities for the permanent shock and for the shock to stochastic volatility are much flatter than for the power utility model, and in particular, the short-term shock price elasticity is relatively large for the permanent shock to consumption.

[12] I exploit the continuous-time quasi-analytical formulas given by Hansen (2012) for the actual computations.

[13] See Hansen (2012) for a discussion of the sensitivity to the parameter ρ, which governs the intertemporal elasticity of substitution.

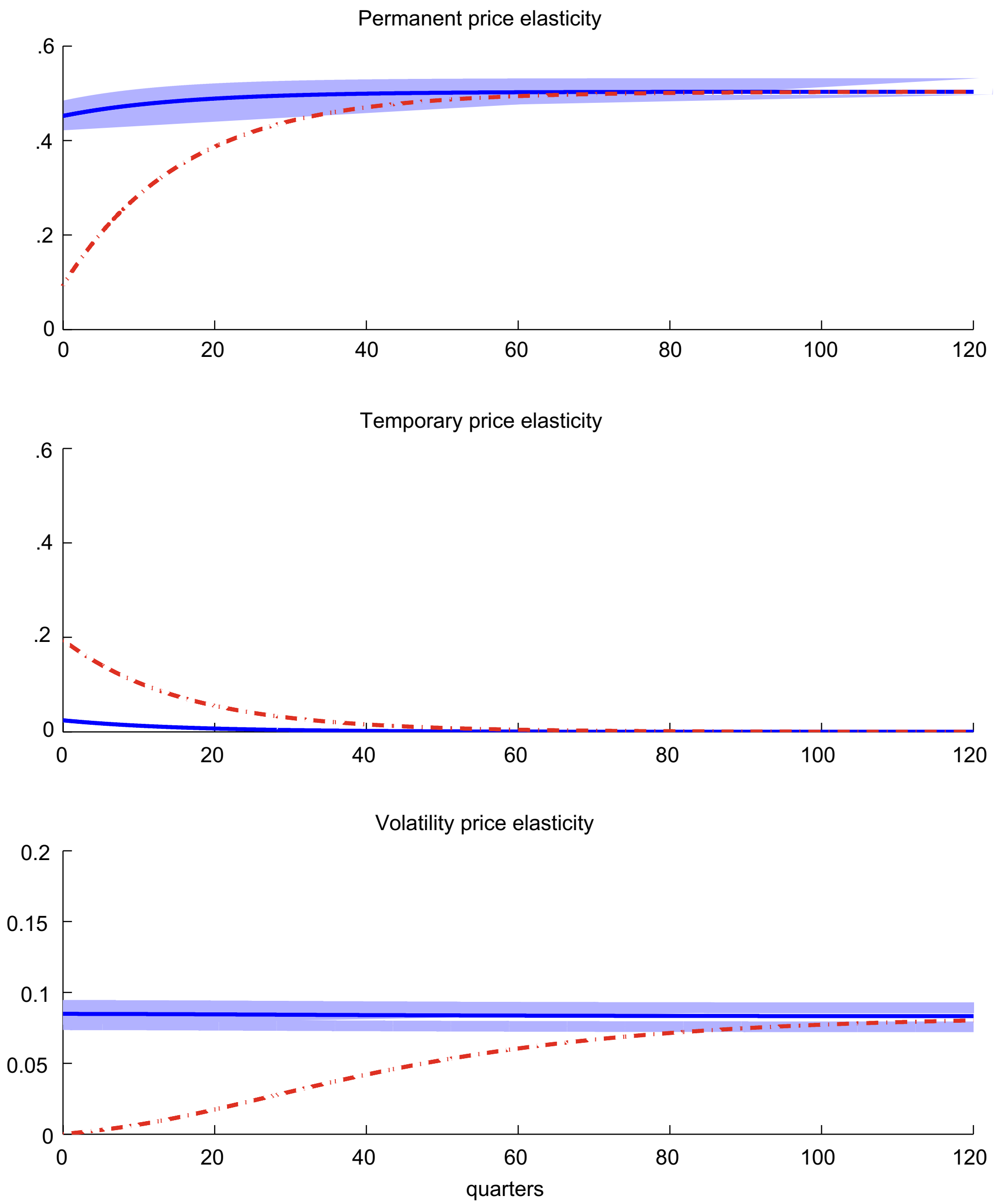

Figure 2 This figure depicts the shock-price elasticities of the three shocks for a model with power utility ($\rho = \gamma = 8$) depicted by the dashed line and with recursive utility ($\rho = 1, \gamma = 8$) depicted by the solid line. The shaded region gives the interquartile range of the shock price elasticities induced by state dependence for the recursive utility model.

The presence of stochastic volatility induces state dependence in all of the shock-price elasticities. This dependence is reflected in the shaded portions in Figure 2 and is of particular interest for the permanent shock, and its presence is a source of time variation in the elasticities for each of the investment horizons.

The amplification of the short-term shock price elasticities has been emphasized at the outset in the literature on "long run risk" through the guises of the recursive utility model. Figure 2 provides a more complete picture of risk pricing. The fact the limiting behavior for recursive and power utility specifications are in agreement follows from the factorization (11).

Models with external habit persistence provide a rather different characterization of shock price elasticities as I will now illustrate.

3.3.3 External Habit Models

Borovička et al. (2011) provide a detailed comparison of the pricing implications of two specifications of external habit persistence, one given in Campbell and Cochrane (1999) and the other in Santos and Veronesi (2006). In order to make the short-term elasticities comparable, Borovička et al. (2011) modified the parameters for the Santos and Veronesi (2006) model. Borovička et al. (2011) performed their calculations using a continuous-time specification in which consumption is a random walk with drift when specified in logarithms. Thus, in contrast to the "long-run risk model", the consumption exposure elasticities are constant:

$$\mathrm{d}\log C_t = .0054\,\mathrm{d}t + .0054\,\mathrm{d}W_t,$$

where W is a scalar standard Brownian motion and the numerical value of μ_c is inconsequential to our calculations. I will not elaborate on the precise construction of the social habit stock used to model the consumption externality and instead posit the implied stochastic discount factors. The constructions differ and are delineated in the respective papers. Rather than embrace a full structural interpretation of the consumption externality, I will focus on the specification of the stochastic discount factors for the two models.

For Santos and Veronesi (2006), the stochastic discount factor is

$$\frac{S_t}{S_0} = \exp(-\delta t)\left(\frac{C_t}{C_0}\right)^{-2}\frac{X_t+1}{X_0+1},$$

where

$$\mathrm{d}X_t = -.035(X_t - 2.335)\mathrm{d}t - .496\,\mathrm{d}W_t.$$

Thus the shock to $\mathrm{d}X_t$ is proportional to the shock to $\mathrm{d}\log C_t$ with the same magnitude but opposite sign. In our calculations we set $G = C$. Consequently, the martingale component of the stochastic discount factor is given by

$$\frac{M_t}{M_0} = \exp\left[(.0054)\,W_t - W_0 - \frac{t}{2}(.0054)^2\right]$$

and the Perron–Frobenius eigenfunction is $e(x) = \frac{1}{x+1}$.

For Campbell and Cochrane (1999), the stochastic discount factor is

$$\frac{S_t}{S_0} = \exp(-\delta t)\left(\frac{C_t}{C_0}\right)^{-2}\frac{\exp(2X_t)}{\exp(2X_0)},$$

where

$$\mathrm{d}X_t = -.035(X_t - .4992) + \left(1 - \sqrt{1 + 1200X_t}\right)\mathrm{d}W_t.$$

In this case the Perron–Frobenius eigenfunction is $e(x) = \exp(-2x)$. The martingale components of S are the same for the two models, as are the martingale components of SG.

Figure 3 depicts the shock-price elasticities for the two models for the quartiles of the state distribution. While the starting points and limit points for the shock-price trajectories agree, there is a substantial difference in how fast the trajectories approach their limits. The long-term limit point is the same as that for a power utility specification ($\rho = \gamma = 2$). For the Santos and Veronesi (2006) specification, the consumption externality is arguably a transient model component. For the Campbell and Cochrane (1999) specification, this externality has very durable pricing implications even if formally speaking this model feature is *transient*. The nonlinearities in the state dynamics apparently compound in a rather different manner for the two specifications. See Borovićka et al. (2011) for a more extensive comparison and discussion.

These examples all feature models with directly specified consumption dynamics. While this has some pedagogical simplicity for comparing impact of investor preferences on asset prices, it is of considerable interest to apply these dynamic value decomposition (DVD) methods to a richer class of economies including economies with multiple capital stocks. For example, Borovicka and Hansen (2012) apply the methods to study a production economy with "tangible" and "intangible" capital as modeled in Ai, Croce, and Li (2010). Richer models will provide scope for analyzing the impact of shock exposures with more interesting economic interpretations.

The elasticities displayed here are local in nature. They feature small changes in exposure to normally distributed shocks. For highly nonlinear models, global alternatives may well have some appeal, or at the very least alternative ways to alter exposure to non-Gaussian tail risk.

4. MARKET RESTRICTIONS

I now explore the stochastic discount factors that emerge from some benchmark economies in which there is imperfect risk sharing. In part, my aim is to provide a characterization of how these economies relate to the more commonly used

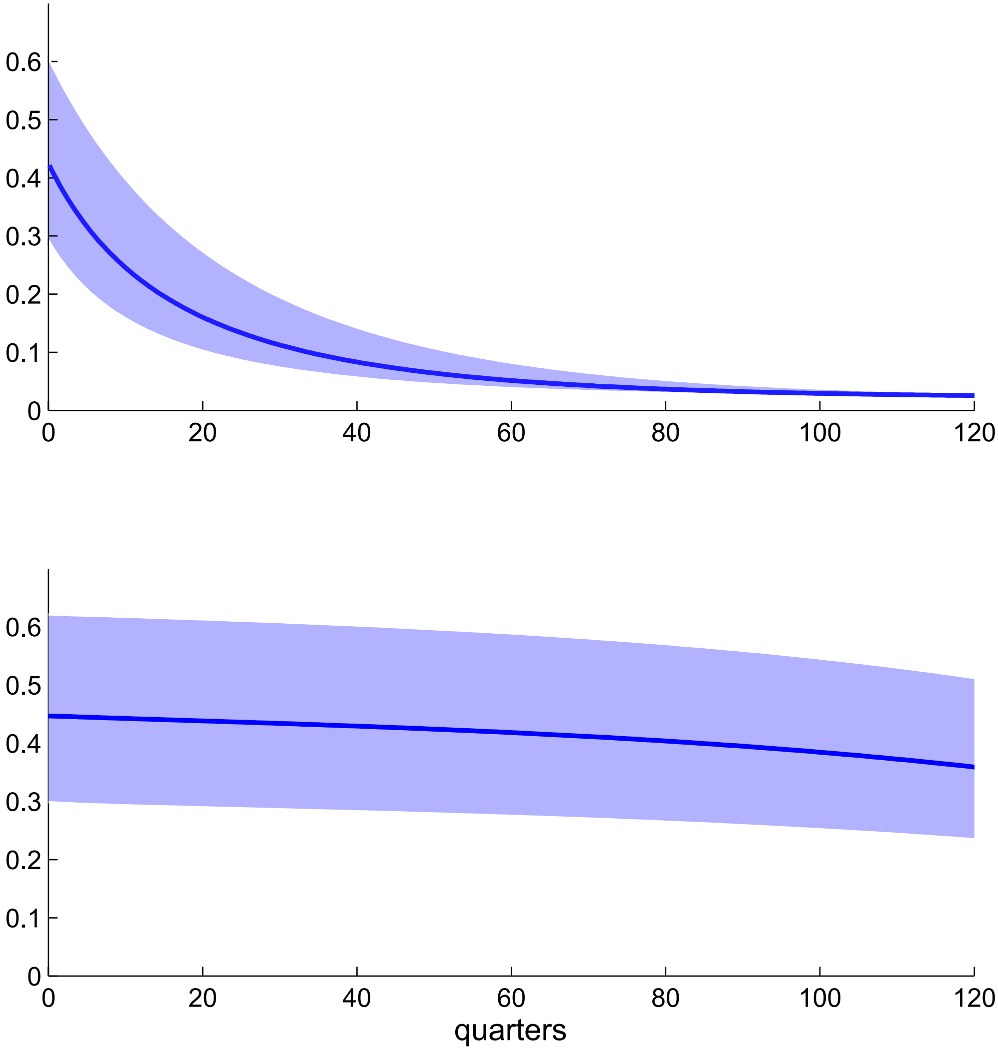

Figure 3 This figure depicts the shock-price elasticities for this single shock specification of the models with consumption externalities. The top panel displays the shock-price elasticity function in the Santos and Veronesi (2006) specification, while the bottom panel displays the Campbell and Cochrane (1999) specification. The solid curve conditions on the median state, while the shaded region depicts the interquartile range induced by state dependence.

structural models of asset pricing. The cross-sectional distribution of consumption matters in these examples, and this presents interesting challenges for empirical implementation. While acknowledging these challenges, my goal is to understand how these distributional impacts are encoded in asset prices over alternative investment horizons.

I study some alternative benchmark economies with equilibrium stochastic discount factor increments that can be expressed as:

$$\frac{S_{t+1}}{S_t} = \left(\frac{S^a_{t+1}}{S^a_t}\right)\left(\frac{S^c_{t+1}}{S^c_t}\right), \tag{16}$$

where the first term on the right-hand side, $\frac{S^a_{t+1}}{S^a_t}$, coincides with that of a representative consumer economy and the second term, $\frac{S^c_{t+1}}{S^c_t}$, depends on the cross-sectional distribution of consumption relative to an average or aggregate. In the examples that I explore,

$$\frac{S^a_{t+1}}{S^a_t} = \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho},$$

where C^a denotes aggregate consumption. The way in which S^c depends on the cross-section differs in the example economies that I discuss, because the market restrictions differ. As in the literature that I discuss, I allow the cross-sectional distribution of consumption (relative to an average) to depend on aggregate states.

While a full characterization of the term structure implications for risk prices is a worthy goal, here I will only initiate such a discussion by investigating when these limits on risk sharing lead to "transient" vs. "permanent" implications for market values. In one case below, $S^c_t = f(X_t)$ for some (Borel measurable) function f of a stochastically stable process X. Thus we know that introducing market imperfections has only transient consequences. For the other examples, I use this method to indicate the sources within the model of the long-term influence of cross-sectional consumption distributions on asset values.

4.1 Incomplete Contracting

Our first two examples are economies in which there are aggregate, public shocks and idiosyncratic, private shocks. Payoffs can be written on the public shocks but not on the private shocks. Let $\mathcal{G}_t$ denote the sigma algebra that includes both public and private shocks, and let $\mathcal{F}_t$ denote the sigma algebra that includes only public shocks. By forming expectations of date t random variables that are $\mathcal{G}_t$ measurable conditioned on $\mathcal{F}_t$, we aggregate over the idiosyncratic shocks but condition on the aggregate shocks. We use this device to form cross-sectional averages. I presume that

$$E(Q_{t+1}|\mathcal{G}_t) = E(Q_{t+1}|\mathcal{F}_t),$$

whenever Q_{t+1} is $\mathcal{F}_t$ measurable. There could be time invariant components to the specification of $\mathcal{G}_t$, components that reflect an individual's type.

In what follows I use C_t to express consumption in a manner that implicitly includes dependence on idiosyncratic shocks. Thus C_t is $\mathcal{G}_t$ measurable. Thus the notation C_t includes a specification of consumption allocated to a cross-section of individuals at date t. With this notation, aggregate consumption is:

$$C_t^a = E(C_t|\mathcal{F}_t). \tag{17}$$

As an example, following Constantinides and Duffie (1996) consider consumption allocations of the type:

$$\log C_{t+1} - \log C_t = \log C_{t+1}^a - \log C_t^a + V_{t+1}Z_{t+1} - \frac{1}{2}(Z_{t+1})^2, \tag{18}$$

where V_{t+1} is $\mathcal{G}_{t+1}$ measurable and a standard normally distributed random variable conditioned on composite event collection: $\mathcal{G}_t \vee \mathcal{F}_{t+1}$. The random variable Z_{t+1} is in the public information set $\mathcal{F}_{t+1}$. It now suffices to define the initial cross-sectional average

$$C_0^a = E[C_0|\mathcal{F}_0].$$

Then (17) is satisfied for other t because

$$E\left(\exp\left[V_{t+1}Z_{t+1} - \frac{1}{2}(Z_{t+1})^2\right]|\mathcal{G}_t \vee \mathcal{F}_{t+1}\right) = 1.$$

In this example, since V_{t+1} is an idiosyncratic shock, the idiosyncratic contribution to aggregate consumption has permanent consequences where the aggregate random variable Z_{t+1} shifts the cross-sectional consumption distribution. Shortly we will discuss a decentralization that accompanies this distribution for which aggregate uncertainty in the cross-sectional distribution of consumption matters for valuation. This is just an example, and more general and primitive starting points are also of interest.

In what follows, to feature the role of market structure we assume a common discounted power utility function for consumers $\rho = \gamma$. The structure of the argument is very similar to that of Kocherlakota and Pistaferri (2009), but there are some differences.[14] Of course one could "add on" a richer collection of models of investor preferences, and for explaining empirical evidence there may be good reason to do so. To exposit the role of market structure, I focus on a particularly simple specification of consumer preferences.

[14] The decentralization of the private information Pareto optimal allocation exploits in a part a derivation provided to me by Fernando Alvarez.

4.1.1 Trading Assets that Depend Only on Aggregate Shocks

First I consider a decentralized economy in which heterogenous consumers trade securities with payoffs that only depend on the aggregate states. Markets are incomplete because consumers cannot trade.

I introduce a random variable Q_{t+1} that is $\mathcal{F}_{t+1}$ measurable. Imagine adding rQ_{t+1} to the $t+1$ consumption utilities. The date t price of the payoff rQ_{t+1} is

$$rE\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}|\mathcal{F}_t\right],$$

which must be subtracted from the date t consumption. The scalar r can be $\mathcal{G}_t$ measurable. We consider an equilibrium allocation for C, and thus part of the equilibrium restriction is that $r=0$ be optimal. This leads to the first-order conditions:

$$(C_t)^{-\rho}E\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}|\mathcal{F}_t\right] = \exp(-\delta)E\left[(C_{t+1})^{-\rho}Q_{t+1}|\mathcal{G}_t\right]. \tag{19}$$

In order to feature the cross-sectional distribution of consumption, I construct:

$$c_t = \frac{C_t}{C_t^a}.$$

I divide both sides by $(C_t^a)^{-\rho}$:

$$(C_t)^{-\rho}E\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}|\mathcal{F}_t\right] = \exp(-\delta)E\left[(c_{t+1})^{-\rho}\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}Q_{t+1}|\mathcal{G}_t\right].$$

I consider two possible ways to represent the stochastic discount factor increment. First divide by the (scaled) marginal utility $(c_t)^{-\rho}$ and apply the Law of Iterated Expectations:

$$E\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}|\mathcal{F}_t\right] = \exp(-\delta)E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}Q_{t+1}|\mathcal{F}_t\right].$$

By allowing trades among assets that include any bounded payoff that is $\mathcal{F}_{t+1}$ measurable, it follows that

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}|\mathcal{F}_{t+1}\right]. \tag{20}$$

See Appendix A. This generalizes the usual power utility model representative agent specification of the one-period stochastic discount factor. Because of the preclusion of trading based on idiosyncratic shocks, investors equate the *conditional expectations* of their intertemporal marginal rates of substitution conditioned only on aggregate shocks. This gives one representation of the limited ability to share risks with this market structure.

For an alternative representation, use the Law of Iterated Expectations on both the left- and right-hand sides of (19) to argue that

$$E[(c_t)^{-\rho}|\mathcal{F}_t]E\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}|\mathcal{F}_t\right]=\exp(-\delta)E\left(E[(c_{t+1})^{-\rho}|\mathcal{F}_{t+1}]\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}Q_{t+1}|\mathcal{F}_t\right).$$

Again I use the flexibility to trade based on aggregate shocks to claim that

$$\frac{S_{t+1}}{S_t}=\exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}\frac{E[(c_{t+1})^{-\rho}|\mathcal{F}_{t+1}]}{E[(c_t)^{-\rho}|\mathcal{F}_t]}. \tag{21}$$

For specification (18) suggested by Constantinides and Duffie (1996), $c_{t+1}-c_t$ is conditionally log-normally distributed and as a consequence,

$$E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}|\mathcal{F}_{t+1}\right]=\exp\left[\frac{\rho(\rho+1)(Z_{t+1})^2}{2}\right].$$

In this special case,

$$\frac{S_{t+1}}{S_t}=\exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho}\exp\left[\frac{\rho(\rho+1)(Z_{t+1})^2}{2}\right].$$

This is just an example, but an informative one. The consumption distribution "fans out" and its dependence on the aggregate state variable Z_{t+1} implies permanent consequences for the stochastic discount factor.[15] There are other mechanisms that might well push against the fanning which are abstracted from in this formulation. For instance, overlapping generations models can induce some reversion depending on how the generations are connected and how new generations are endowed.

4.1.2 Efficient Allocations with Private Information

One explicit rationale for limiting contracting to aggregate shocks is that idiosyncratic shocks reflect private information. In an interesting contribution, Kocherlakota and Pistaferri (2007, 2009) propose a decentralization of constrained efficient allocations represented via the construction of a stochastic discount factor. Kocherlakota and Pistaferri consider the case of constraint efficient allocations where agents' preferences

[15] In the degenerate case in which Z is constant over time, the impact of the cross-sectional distribution will only be to scale the stochastic discount factor and hence prices will be scaled by a common factor. Risk and shock-price elasticities will coincide with those from the corresponding representative consumer model.

are given by expected discounted utility with an additive sub-utility of consumption and leisure (or effort), and where the consumption sub-utility is specified as a power utility function, $\rho = \gamma$. Individual agents' leisure (effort) needed to produce a given output is private information. Individuals cannot hide consumption, however, through even inefficient storage.

Kocherlakota and Pistaferri (2009) take as given the solution of planning a problem where agent's effort is unobservable. How this efficient allocation is attained is an interesting question in its own right, a question that is of direct interest and discussed extensively in the literature on contracting in the presence of private information. To decentralize these allocations, Kocherlakota and Pistaferri consider intermediaries that can observe the consumption of the agents and that can trade among themselves. They distinguish between aggregate shocks (which are public) and idiosyncratic shocks (which are private but diversifiable). As with the incomplete financial market model that I discussed previously. Intermediaries trade among themselves in complete markets on all public shocks and engage a large number of agents so they completely diversify the privately observed shocks. The contract of the intermediary with the agents ensures that the reports are correct. The objective of the intermediaries is to minimize the cost, at market prices, of delivering agents a given lifetime utility. This intermediary provides a way to deduce the corresponding stochastic discount factor for assigning values to payoffs on the aggregate state.

I introduce a random variable Q_{t+1} that is $\mathcal{F}_{t+1}$ measurable. Imagine adding rQ_{t+1} to the $t+1$ period utilities instead to the period $t+1$ consumption. Due to the additive separability of the period utility function adding an amount of *utils* both on t and across all continuations at $t+1$ does change the incentives for the choice of leisure (effort). This leads me to consider the equivalent adjustment $\Delta_{t+1}(r)$ to consumption:

$$rQ_{t+1} + U(C_{t+1}) = U\left[C_{t+1} + \Delta_{t+1}(r)\right].$$

I have altered the $t+1$ period cross-sectional utility in a way that is equivalent to changing the utility to the efficient allocation of consumption in the cross-sectional distribution at date $t+1$ in a manner that does not depend on the idiosyncratic shocks. To support this change, however, the change in consumption $\Delta_{t+1}(r)$ does depend on idiosyncratic shocks. Differentiating with respect to r:

$$Q_{t+1} = (C_{t+1})^{\rho} \left. \frac{\Delta_{t+1}}{\mathrm{d}r} \right|_{r=0}.$$

Thus

$$\left. \frac{\mathrm{d}\Delta_{t+1}}{\mathrm{d}r} \right|_{r=0} = Q_{t+1}(C_{t+1})^{\rho}.$$

To compensate for the rQ_{t+1} change in the next period (date $t+1$) utility, subtract

$$\exp(-\delta)E(Q_{t+1}|\mathcal{G}_t) = \exp(-\delta)E(Q_{t+1}|\mathcal{G}_t)$$

from the current (date t) utility. This leads me to solve:

$$-r\exp(-\delta)E(Q_{t+1}|\mathcal{F}_t) + U(C_t) = U[(C_t) - \Theta_t(r)].$$

Again differentiating with respect to r,

$$\exp(-\delta)E(Q_{t+1}|\mathcal{F}_t) = (C_t)^{-\rho}\left.\frac{\mathrm{d}\Theta_t}{\mathrm{d}r}\right|_{r=0},$$

or

$$\left.\frac{\mathrm{d}\Theta_t}{\mathrm{d}r}\right|_{r=0} = \exp(-\delta)E(Q_{t+1}|\mathcal{F}_t)(C_t)^{\rho}.$$

The members of our family of rQ_{t+1} perturbations have the same continuation values as those in the efficient allocation. By design the perturbations are equivalent to a transfer of utility across time periods that does not depend on idiosyncratic shocks. These two calculations are inputs into first-order conditions for the financial intermediary.

The financial intermediary solves a cost minimization problem:

$$\min_r -E[\Theta_t(r)|\mathcal{F}_t] + E\left[\left(\frac{S_{t+1}}{S_t}\right)\Delta_{t+1}(r)|\mathcal{F}_t\right].$$

We want the minimizing solution to occur when r is set to zero. The first-order conditions are:

$$-E\left[\left.\frac{\mathrm{d}\Theta_t}{\mathrm{d}r}\right|_{r=0}|\mathcal{F}_t\right] + E\left[\left(\frac{S_{t+1}}{S_t}\right)\left.\frac{\mathrm{d}\Delta_{t+1}}{(\mathrm{d}r)}\right|_{r=0}|\mathcal{F}_t\right] = 0.$$

Substituting for the Δ_t and Θ_t derivatives,

$$-\exp(-\delta)E(Q_{t+1}|\mathcal{F}_t)E[(C_t)^{\rho}|\mathcal{F}_t] + E\left[\left(\frac{S_{t+1}}{S_t}\right)Q_{t+1}E[(C_{t+1})^{\rho}|\mathcal{F}_{t+1}]\mathcal{F}_t\right] = 0. \tag{22}$$

Let

$$D_{t+1} = \exp(\delta)\left(\frac{S_{t+1}}{S_t}\right)\frac{(C^a_{t+1})^{\rho}\,[(c_{t+1})^{\rho}|\mathcal{F}_{t+1}]}{(C^a_t)^{\rho}E\,[(c_t)^{\rho}|\mathcal{F}_t]},$$

where I have used the fact that the cross-sectional averages C_t^a and C_{t+1}^a are in the respective information sets of aggregate variables. Then

$$E(Q_{t+1}|\mathcal{F}_t) = E[D_{t+1}Q_{t+1}|\mathcal{F}_t].$$

Given flexibility in the choice of the $\mathcal{F}_{t+1}$ measurable random variable Q_{t+1}, I show in Appendix A that $D_{t+1} = 1$, giving rise to the "inverse Euler equation":

$$\left(\frac{S_{t+1}}{S_t}\right) = \exp(-\delta)\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho} \frac{E[(c_t)^\rho|\mathcal{F}_t}{E[(c_{t+1})^\rho|\mathcal{F}_{t+1}} \tag{23}$$

suggested by Kocherlakota and Pistaferri (2009). The "inverse" nature of the Euler equation emerges because my use of utility-based perturbations is based on aggregate shocks rather than direct consumption-based perturbations. This type of Euler equation is familiar from the seminal work of Rogerson (1985).

An alternative, but complementary analysis derives the full solution to the constraint efficient allocation. At least since the work of Atkeson and Lucas (1992), it is known that even temporary idiosyncratic shocks create a persistent trend in dispersion of consumption. Hence this particular way of modeling private information has the potential of important effects on the long-term (martingale) component to valuation.

In the incomplete contracting framework we were led to consider the time series of cross-sectional moments $\{E[(c_t)^{-\rho}|\mathcal{F}_t] : t = 1, 2, \ldots\}$, whereas in this private information, Pareto-efficient economy we are led to consider $\{E[(c_t)^\rho|\mathcal{F}_t] : t = 1, 2, \ldots\}$. These two models feature rather different attributes, including tails behavior of the cross-sectional distribution for consumption. Both, however, suggest the possibility of long-term contributions to valuation because of the dependence of the cross-sectional distribution on economic aggregates. There are important measurement challenges that arise in exploring the empirical underpinnings of these models, but some valuable initial steps have been taken by Brav, Constantinides, and Geczy (2002), Cogley (2002), and Kocherlakota and Pistaferri (2009).

4.2 Solvency Constraints

In this section I discuss the representation of a stochastic discount factor in models where agents occasionally face binding solvency constraints. One tractable class of models that features incomplete risk sharing is one where agents have access to complete markets but where the total value of their financial wealth is constrained (from below) in a state contingent manner. Following Luttmer (1992, 1996), and He and Modest (1995), I refer to such constraints as solvency constraints. In contrast to the models with incomplete contracting based on information constraints, I no longer distinguish between $\mathcal{G}_t$ and $\mathcal{F}_t$; but I do allow for some ex ante heterogeneity in endowments or labor income. Suppose there are i types of investors, each with consumption C_t^i. Investor types may have different initial asset holdings and may have different labor income or endowment processes. Let C_t^a denote the average across all consumers and $c_t^i = \frac{C_t^i}{C_t^a}$. Under expected discounted utility preferences with a power specification $(\rho = \gamma)$, the stochastic discount factor increment is:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho} \max_i \left\{\left(\frac{c_{t+1}^i}{c_t^i}\right)^{-\rho}\right\}. \tag{24}$$

To better understand the origin of this formula, notice that an implication of it is:

$$\frac{S_{t+1}}{S_t} \geqslant \exp(-\delta)\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\rho}, \tag{25}$$

which is featured in the work Luttmer (1992, 1996) and He and Modest (1995).

To better understand this inequality, observe that positive scalar multiples $r \geqslant 0$ of a positive payoff $Q_{t+1} \geqslant 0$ when added to composite equilibrium portfolio payoff of person i at date $t + 1$ will continue to satisfy the solvency constraint. Thus such a perturbation is an admissible one. When I optimize with respect to r, I now impose the constraint that $r \geqslant 0$; and this introduces a Kuhn–Tucker multiplier into the calculation. The first-order condition for r is

$$(C_t^i)^{-\rho} E\left[\left(\frac{S_{t+1}}{S_t}\right) Q_{t+1} | \mathcal{F}_t\right] \geqslant \exp(-\delta) E[(C_{t+1}^i)^{-\rho} Q_{t+1} | \mathcal{F}_t],$$

where the inequality is included in case the nonnegativity constraint on r is binding. Since this inequality is true for any bounded, positive, nonnegative $\mathcal{F}_{t+1}$ measurable payoff Q_{t+1}, inequality relation (25) holds.

Formula (24) is a stronger restriction and follows since equilibrium prices are determined by having at least one individual that is unconstrained in the different realized date $t + 1$ states of the world. The max operator captures the feature that the types with the highest valuation are unconstrained. While in principle expression (24) can be estimated using an empirical counterpart to the type's i consumption, the presence of the max in conjunction with error-ridden data makes the measurement daunting.

Luttmer (1992) takes a different approach by exploiting the implications via aggregation of solvency constraints and analyzing the resulting inequality restriction.[16] This same argument is revealing for my purposes as well. Since ρ is positive, inequality (25) implies that

$$(C_t^i)\left(\frac{S_{t+1}}{S_t}\right)^{-\frac{1}{\rho}} \leqslant \exp(-\delta) C_{t+1}^i$$

[16] See Hansen, Heaton, and Luttmer (1995) for a discussion of econometric methods that support such an approach.

where the inequality is reversed because $-\frac{1}{\rho} < 0$. Forming a cross-sectional average, preserves the inequality:

$$(C_t^a)\left[\exp(\delta)\frac{S_{t+1}}{S_t}\right]^{-\frac{1}{\rho}} \leqslant C_{t+1}^a.$$

Raising both sides to the negative power $-\rho$ reverses again the inequality:

$$(C_t^a)^{-\rho}\left[\exp(\delta)\frac{S_{t+1}}{S_t}\right] \geqslant (C_{t+1}^a)^{-\rho}.$$

Rearranging terms gives the inequality of interest

$$\frac{S_{t+1}}{S_t} \geqslant \exp -\delta\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}. \tag{26}$$

From this inequality, we can rule out any hope that solvency-constraint models only have transient consequences for valuation because we cannot hope to write

$$\frac{S_{t+1}}{S_t} = \exp -\delta\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}\left[\frac{h(X_{t+1})}{h(X_t)}\right]$$

for some h and some stochastically stable process X, unless of course $h(X)$ is constant with probability one. In the degenerate case, the solvency constraints are not binding. At the very least the interest rates on discount bonds, including the long-term interest rate, have to be smaller than those in the corresponding representative consumer economy.

Alvarez and Jermann (2000, 2001) and Chien and Lustig (2010) impose more structure on the economic environment in order to get sharper predictions about the consumption allocations. They use limited commitment as a device to set the solvency thresholds needed to compute an equilibrium.[17] Following Kehoe and Levine (1993), these authors introduce an outside option that becomes operative if an investor defaults on the financial obligations. The threat of the outside option determines the level of the solvency constraint that is imposed in a financial market decentralization. Solvency constraints are chosen to prevent that the utility value to staying in a market risk sharing arrangement to be at least as high as the utility of the corresponding outside options. Alvarez and Jermann (2001) and Chien and

[17] Zhang (1997) considers a related environment in which borrowing constraints are endogenously determined as implication of threats to default. Alvarez and Jermann (2000), Alvarez and Jermann (2001) and Chien and Lustig (2010) extend Zhang (1997) by introducing a richer collection of security markets.

Lustig (2010) differ in the precise natures of the market exclusions that occur when investors walk away from their financial market obligations. Alvarez and Jermann (2000) argue that the cross-sectional distribution in example economies with solvency constraints stable in the sense that $C_t^i = h_i(X_t)C_t^a$ for some stochastically stable Markov process X. Thus

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho} \max_i \left\{\left(\frac{h_i(X_{t+1})}{h_i(X_t)}\right)^{-\rho}\right\}.$$

Notice that the objective of the max operation is a ratio of a common function of the Markov state over adjacent periods. Even so, given our previous argument the outcome of this maximization will not have an expression as an analogous ratio. Even with a stable consumption allocation, the presence of solvency constraints justified by limited commitment may have long-term consequences for valuation.[18]

Chien and Lustig (2010) also provide a suggestive characterization for the stochastic discount factor ratio of the form:

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{C_{t+1}^a}{C_t^a}\right)^{-\rho}\left(\frac{Z_{t+1}^a}{Z_t}\right)^{-\rho}.$$

Thus $(Z_t)^\rho = S_t^c$ in representation (16). Chien and Lustig interpret Z and characterize the ratio $\frac{Z_{t+1}}{Z_t}$ using numerical methods. In particular, they show that the positive process Z does not decrease over time.[19] This property for Z is to be anticipated from (26).

[18] Alvarez and Jermann (2000) derive a different type of factorization of a stochastic discount factor under some very special restrictions. They express the endowments for each investor type as a product of the aggregate endowment and a share of that endowment. They suppose that the growth in the aggregate endowment is itself independent and identically distributed and that the aggregate endowment process is independent of the vector of endowment share processes. They argue that equilibrium stochastic discount factor is the product of the corresponding representative consumer stochastic discount factor and a term that is independent of the process for aggregate endowment growth. (See the proof of their Proposition 5.4.) Thus the two terms in factorization (16) are statistically independent in their example. The prices of payoffs that depend only on aggregate endowment growth process over a fixed investment horizon have a common term that emerges because of the contribution of the share process history to the stochastic discount factor. By forming price ratios, this share process contribution to valuation is netted out. For a cash flow G that depends only on the aggregate endowment and not on the share process, the cash-flow risk premia as measured by formula (15) will be the same as for the corresponding representative consumer model with iid consumption growth. An analogous simplification applies to payoffs that depend only on share process and not on the aggregate endowment.

[19] See also Alvarez and Jermann (2000) Proposition 5.2 for the analogous result for the limited commitment economies that they study.

Unless it is degenerate, such a process cannot be stationary, but its logarithm can have stationary increments, as presumed in my analysis.

Even though the consumption distribution may not "fan out" over time, evidently the introduction of solvency constraints has important long-term consequences for valuation. A remaining challenge is to understand better when the implied state dependence in the cross-sectional distribution of individual consumption ratios induces changes in the long-term risk prices.

4.3 Segmented Market and Nominal Shocks

In this section I explore environments that feature both nominal shocks and segmented asset markets. The transient consequences of nominal shocks have been featured in other environments. For instance in log-linear specifications of the macro time series, Blanchard and Quah (1989) and Shapiro and Watson (1988) use the transient nature of nominal shocks as a device to identify transient and permanent sources of economic fluctuations. Here I explore economic models with explicit transition mechanisms to investigate further the transient nature of nominal shocks for valuation including adjustments for risk exposure.

As an illustration, I consider the stylized model of Alvarez, Atkeson, and Kehoe (2002, 2009) where both nominal cash flows and segmentation are introduced. For simplicity, these models presume a binding cash-in-advance constraint, and hence the price level is proportional to money supply. Consumers can transfer cash between their "brokerage account" (where an intermediary with access to complete security markets manages their portfolio) and a liquid asset that must be used for consumption expenditures. For consumers to embark on this transfer, they must pay a fixed cost. If they decide not to pay the fixed cost, they must consume the real value of the accumulated nominal income. In equilibrium only some of the consumers participate in asset markets, but those that do so have the same consumption, which I denote by C_t^p. In this case the one-period stochastic discount factor is given by the corresponding one-period intertemporal marginal rate of substitution of the participants adjusted for changes in the nominal price level

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{P_t}{P_{t+1}}\right)\left(\frac{C_{t+1}^p}{C_t^p}\right)^{-\rho},$$

where P_t is the date t price level and where S_t is the nominal discount factor. This formula is true even though the identity of the participants changes over time. The equilibrium of this model is such that

$$C_p^t = C_p^t h_p(\log m_t - \log m_{t-1}),$$

where $\log m_t - \log m_{t-1}$ is the growth rate of the money supply and C_t^a is aggregate consumption. Thus

$$\frac{S_{t+1}}{S_t} = \exp(-\delta)\left(\frac{P_t}{P_{t+1}}\right)\left(\frac{C_{t+1}^p}{C_t^p}\right)^{-\rho}\left[\frac{h(\log m_{t+1} - \log m_t)}{h(\log m_t - \log m_{t-1})}\right], \tag{27}$$

where $h = (h_p)^{-\rho}$.

In transaction cost models such as this one, the impact on risk pricing is *transient* relative to a standard power utility model. The impact of changes in participation is captured by the function h of the presumed stationary growth rate in the money supply. Nominal fluctuations influence real outcomes. The factorization in (27) features both standard nominal effects on valuation from both the nominal price level P, and aggregate consumption C^a. Both contribute to the martingale component of the stochastic discount factor S. This type of model can and has been used to study both term premium for nominal interest rates as well as for nominal exchange rates. Third, one can use the growth decomposition described in (13) by letting $G = P$ to distinguish real cash flows from nominal ones, or more generally by letting a multiplicative component of G be P as a device to take nominal growth into account for computing real risk-price and shock-price elasticities induced by nominal shocks.

In this section I have only "scratched the surface" so to speak in characterizing how and when market restrictions alter the term structure of shock and risk price elasticities through its implications for the time series behavior of cross-sectional distributions. As a starting point I have examined when the long-term discount rate and the martingale components of stochastic discount factors are altered, but as I argued earlier in a different context this is merely a starting point for a more complete analysis.

5. CONCLUSIONS

In this chapter I have focused on characterizing asset values through the lens of economic models of valuation. By using structural models, models with an explicitly specified preference and market structure, researchers can assign values to a rich collection of cash flows and risk prices for the exposures to alternative underlying macroeconomic shocks. These DVD methods that I discussed allow researchers to extract pricing implications for cash flows without resorting to log-linearization. I consider the global "entropy" methods and local "elasticity" methods based on perturbing the exposure of cash flows to shocks as complementary devices to characterize the sensitivity of risk prices to the investment horizon. The DVD methods are supported in part by a factorization that provides a mathematical formalization of permanent and transitory components to valuation. The distinction between permanent and transitory rests formally on limiting behavior when we extend the investment horizon. As with related time series decompositions, this distinction enhances both model building and testing by clarifying when the permanent and transitory distinction is sharp and when it is blurred.

While this is a chapter about models, it is also suggestive of what hypothetical securities might be most useful in distinguishing among competing models. As time series data on richer collections of equity-based derivative contracts become available, they offer the promise to pose direct challenges to the underlying pricing of cash flows. Complementary econometric and empirical advances will enhance our understanding of the empirical underpinnings of structural models of asset prices.

APPENDIX A. LIMITED CONTRACTING ECONOMIES REVISITED

In this appendix I complete two of the arguments made in Section 4.1. Consider first an argument in Section 4.1.1. There I showed that

$$E\left[\left(\frac{S_{t+1}}{S_t}\right) Q_{t+1}|\mathcal{F}_t\right] = \exp(-\delta)E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho} Q_{t+1}|\mathcal{F}_t\right]. \tag{28}$$

My aim is to show that formula (20) is satisfied. For convenience, I rewrite it:

$$\left(\frac{S_{t+1}}{S_t}\right) = \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho} E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}|\mathcal{F}_{t+1}\right].$$

Let

$$Q_{t+1} = \begin{cases} 1 & \left(\frac{S_{t+1}}{S_t}\right) > \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho} E\left[\left(\frac{c_{t+1}}{c_t}\right)^{\rho}|\mathcal{F}_{t+1}\right], \\ o & \text{otherwise.} \end{cases}$$

Then it follows from (28) that

$$\left(\frac{S_{t+1}}{S_t}\right) > \exp(-\delta)\left(\frac{C^a_{t+1}}{C^a_t}\right)^{-\rho} E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}|\mathcal{F}_{t+1}\right]$$

with probability zero. An entirely similar argument implies the reverse inequality. Thus representation (20) is valid. Since a similar argument proves (21), I do not repeat the logic.

Consider next an argument in Section 4.1.2. Recall the construction:

$$D_{t+1} = \left(\frac{S_{t+1}}{S_t}\right)\frac{\exp(\delta)E[(C_{t+1})^{\rho}|\mathcal{F}_{t+1}]}{E[(C_t)^{\rho}|\mathcal{F}_t]}$$

and the implication

$$E(D_{t+1}Q_{t+1}|\mathcal{F}_t) = E(Q_{t+1}|\mathcal{F}_t).$$

I now show that by exploiting the flexibility in the choice of $Z_{t+1}, D_{t+1} = 1$ giving rise to the inverse Euler equation: (23). Let

$$Q_{t+1} = \begin{cases} D_{t+1} & \text{if } \mathsf{b} > D_{t+1} > 1, \\ 0 & \text{otherwise.} \end{cases}$$

I impose an upper bound on Q_{t+1} to ensure that perturbation in the date $t+1$ utility can be implemented by choices of individual consumptions. It follows that $\Pr\{\mathsf{b} \geqslant Q_{t+1} > 1\} = 0$. Since b is arbitrary, it must also be true that $\Pr\{D_{t+1} > 1\} = 0$. Similarly, form

$$Q_{t+1} = \begin{cases} 0 & \text{if } D_{t+1} \geqslant 1, \\ D_t & \text{otherwise.} \end{cases}$$

It then follows that $\Pr\{D_{t+1} < 1\} = 0$. Thus $D_{t+1} = 1$.

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