Robust Identification of Investor Beliefs

Lars Peter Hansen
University of Chicago
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collaborators: Xiaohong Chen and Peter G. Hansen

Motivation

Behavioral "distortions" and "ambiguity aversion" are more compelling in environments for which uncertainty is complex and speculation about the future is challenging

- WHAT?
  - We propose and justify a data and model-based method for deducing market beliefs
  - We construct bounds on expectations of unknown future aggregates captured as a nonlinear expectation
- WHY?
  - They provide a formal way to address the public and private sector interest in market perceptions
  - They serve as a diagnostic for models in which asset prices are represented with distorted beliefs

Two observations

Asset prices:

- are forward-looking and serve as barometers for market beliefs
- entangle beliefs and risk aversion

Empirical risk compensation

conditioning	expected return	volatility	"risk" premia
low D/P	6.54 %	.84	5.78
mid D/P	4.70 %	.76	3.40
high D/P	15.41 %	.87	12.37
none	8.94 %	.83	7.22

Two approaches

We could

- impose rational expectations and explore "exotic" or "ad hoc" models with time-varying risk aversion
- model beliefs that are distorted (relative to rational expectations) justified by a) psychology or b) ambiguity aversion with risk aversion constrained to be moderate

We speak to this second approach.

Our approach

- We presume that a dynamic model is misspecified under rational expectations
- To correct this misspecification, we allow for beliefs to differ and to be “distorted” (different from rational expectations)
- We limit the alternative probabilities using statistical measures of “discrepancy” that capture the magnitude of the distortion
- We formally derive bounds on the beliefs that are consistent with the observed asset prices and survey evidence

Related applied math literatures

- generalized empirical likelihood methods
- large deviation theory

We build on some of their insights, but there is an important difference.

> *We alter the beliefs of agents inside the model subject to bounds that limit how much these beliefs conflict with the probabilities implied by historical evidence*

Basic formulation

- Moment equations under rational expectations:

$$\mathbb{E}\left[f(X,\theta) \mid \mathfrak{I}\right] = 0.$$

where the function f captures the parameter dependence (θ) along with variables (X) observed by the econometrician

- A typical asset pricing example:

$$\mathbb{E}(SR - \mathbf{1}_n | \mathfrak{I}) = 0$$

where R is a vector of returns, S is the stochastic discount factor (SDF), $\mathfrak{I}$ denote the investor information set.

For simplicity, I will drop the parameter dependence but comment later on how unknown parameters can be included

Market beliefs

We consider moment restrictions of the form:

$$\mathbb{E}\left[Mf(X) \mid \mathfrak{I}\right] = 0.$$

where $M \geq 0$ and $\mathbb{E}\left(M \mid \mathfrak{I}\right) = 1$.

The random variable M provides a flexible change in the probability measure. M defines a relative density that informs of how the rational expectations are altered by market beliefs.

- We call each M a “belief distortion”
- M not uniquely identified!

Three applications

- risk-neutral pricing

$$S = (R^f)^{-1}$$

where R^f is the one-period risk free rate

- long-term risk-neutral pricing

$$S = (R^h)^{-1}$$

where R^h is the limiting holding period return on a long-term bond

- unitary relative risk aversion in recursive utility

$$S = (R^w)^{-1}$$

where R^w is the one-period return on the wealth portfolio

Quantifying belief distortions

Consider a family of *f*-divergences.

- Introduce a convex function ϕ defined on $\mathbb{R}^+$ for which $\phi(1) = 0$. By Jensen's inequality,

$$\mathbb{E}[\phi(M)] \geq \phi(1) = 0$$

- Special cases include:

  i) $\phi(m) = -\log m$
  ii) $\phi(m) = 4\,(1 - \sqrt{m})$
  iii) $\phi(m) = m \log m$
  iv) $\phi(m) = \frac{1}{2}(m^2 - m)$.

First two examples are problematic for detecting misspecification.
First and third examples have the nicest dynamic extensions.

Bounding expectations

Given a function g of X, solve

$$\mathbb{K}(g) \doteq \min_{M \geq 0} \mathbb{E}\left[Mg(X)\right]$$

subject to the three constraints:

$$\begin{aligned} \mathbb{E}\left[M \log M\right] &\leq \kappa \\ \mathbb{E}\left[Mf(X)\right] &= 0, \\ \mathbb{E}\left[M\right] &= 1. \end{aligned}$$

Using convex duality.

$$\sup_{\xi > 0} \max_{\lambda} -\xi \log \mathbb{E}\left(\exp\left[-\frac{1}{\xi} g(X) - \lambda \cdot f(X)\right]\right) - \xi\kappa$$

Computation

i) for a fixed ξ, solve

$$\max_{\lambda} -\xi \log \mathbb{E}\left(\exp\left[-\frac{1}{\xi}g(X) - \lambda \cdot f(X)\right]\right)$$

ii) compute lower bound on $\underline{\kappa}$

$$\max_{\lambda} - \log \mathbb{E}\left(\exp\left[-\lambda \cdot f(X)\right]\right)$$

ii) choose ξ to hit the relative entropy constraint $\kappa \geq \underline{\kappa}$

iii) calculate the belief distortion:

$$M^{*} = \frac{\exp\left[-\frac{1}{\xi^{*}}g(X) - \lambda^{*} \cdot f(X)\right]}{\mathbb{E}\left(\exp\left[-\frac{1}{\xi^{*}}g(X) - \lambda^{*} \cdot f(X)\right]\right)}$$

Nonlinear expectation

We represent belief distortions as alternative expectations. $\mathbb{K}$ is "nonlinear expectation" mapping bounded functions into real numbers satisfying:

i) if $g_2 \geq g_1$, then $\mathbb{K}(g_2) \geq \mathbb{K}(g_1)$.

ii) if g constant, then $\mathbb{K}(g) = g$.

iii) $\mathbb{K}(\mathsf{r}g) = \mathsf{r}\mathbb{K}(g), \quad \mathsf{r} \geq 0$

iv) $\mathbb{K}(g_1) + \mathbb{K}(g_2) \leq \mathbb{K}(g_1 + g_2)$

Dynamic formulation

- Use conditioning to factor:

$$M = N\mathbb{E}\left(M \mid \mathfrak{I}\right)$$

  where $N = \frac{M}{\mathbb{E}(M|\mathfrak{I})}$ distorts the transition probabilities and $\mathbb{E}\left(M \mid \mathfrak{I}\right)$ distorts the probabilities over the conditioning information.

- In a dynamic setting the two components on the right-hand side are linked.

We now extend the previous approach to account for this connection.

Dynamic recursive formulation

- Environment: Baseline probability triple $(\Omega, \mathfrak{G}, \mathsf{P})$ and a one-to-one transformation $\mathbb{S}$ on Ω which is measure-preserving and ergodic under P.
- Information: Let $\mathfrak{I}_0 \subset \mathfrak{G}$. Associated with the transformation $\mathbb{S}$, let

$$\mathfrak{I}_1 = \left\{\Lambda \in \mathfrak{G} : \mathbb{S}^{-1}\Lambda \in \mathfrak{I}_0\right\},$$

  and define $\mathfrak{F}_t$ analogously for all t. Information accumulates:

$$\mathfrak{I}_0 \subset \mathfrak{I}_1,$$

- Stochastic processes: For any random variable B_0 that is $\mathfrak{I}_0$ measurable, we form $B_t = B_0 \circ \mathbb{S}^t$ in constructing a stochastic process $\{B_t : t \geq 0\}$.

Alternative probabilities

- Let Q denote an alternative probability distribution on $(\Omega, \mathfrak{G})$ that is measure-preserving and ergodic
- Let Q_t be the restriction of Q_t be the restriction of Q to $\mathfrak{I}_t$. We consider only $\mathsf{Q}'s$ for which there exists an $N_1 \geq 0$ that is $\mathfrak{I}_1$ measurable that satisfies:

$$\int B_1 d\mathsf{Q}_1 = \int \mathbb{E}\left(N_1 B_0 \mid \mathfrak{I}_0\right) d\mathsf{Q}_0$$

  for bounded $\mathfrak{I}_0$ measurable random variables B_0.
- Form

$$M_T = \prod_{t=1}^{T} N_t.$$

  Then under Q, the date T conditional expectation is

$$\mathbb{E}\left(M_T B_T \mid \mathfrak{I}_0\right)$$

N_t distorts the one-period transition probabilities between dates $t-1$ and t.

Relative entropy

- Fixed T

$$\mathbb{E}\left(M_T \log M_T \mid \mathfrak{I}_0\right) = \mathbb{E}\left(M_T \sum_{t=1}^{T} \log N_t \mid \mathfrak{I}_0\right) \geq 0$$

- Limiting version:

$$\begin{aligned}
\mathcal{R}(N_1) &= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left(M_T \log M_T \mid \mathfrak{I}_0\right) \\
&= \lim_{T\to\infty} \mathbb{E}\left(M_T \left[\frac{1}{T}\sum_{t=1}^{T} \log N_t\right] \mid \mathfrak{I}_0\right) \\
&= \int \mathbb{E}\left(N_1 \log N_1 \mid \mathfrak{I}_0\right) d\mathsf{Q}_0,
\end{aligned}$$

Relative entropy continued

Two observations about

$$\mathcal{R}(N_1) = \lim_{T \to \infty} \mathbb{E} \left(M_T \left[\frac{1}{T} \sum_{t=1}^{T} \log N_t \right] \mid \mathfrak{I}_0 \right)$$

- division by T allows for the distortions to change the Law of Large Numbers limits
- this measure is central to large deviation assessments of how difficult it is to distinguish alternative probabilities Q (say the possible ones used by investors) from P (the probability that governs the data evolution)

Bounding expectations recursively

Functional equation

$$\mu = \min_{N_1} \mathbb{E}\left(N_1\left[g(X_1) + \xi \log N_1 + v_1\right] | \mathfrak{I}_0\right) - v_0$$

subject to constraints:

$$\mathbb{E}\left[N_1 f(X_1) \mid \mathfrak{I}_0\right] = 0$$
$$\mathbb{E}\left[N_1 \mid \mathfrak{I}_0\right] = 1$$

where $v_1(\omega) = v_0[\mathbb{S}(\omega)]$ and v_0 is $\mathfrak{I}_0$ measurable and μ is a real number. This equation determines the constant μ and the random variable v_0 up to a translation by a constant.

Functional equation revisited

$$\mu = \min_{N_1} \mathbb{E}\left(N_1\left[g(X_1) + \xi \log N_1 + v_1\right] | \mathfrak{I}_0\right) - v_0$$

subject to constraints.

Observation:

$$\begin{aligned}\mu^* =& \xi \lim_{T\to\infty} \mathbb{E}\left(M_T^*\left[\frac{1}{T}\sum_{t=1}^{T} \log N_t^*\right] \mid \mathfrak{I}_0\right) \\ &+ \lim_{T\to\infty} \mathbb{E}\left(M_T^*\left[\frac{1}{T}\sum_{t=1}^{T} g(X_t)\right] \mid \mathfrak{I}_0\right) \\ =& \int \mathbb{E}\left(N_1^*\left[g(X_1) + \xi \log N_1^*\right] | \mathfrak{I}_0\right) d\mathsf{Q}_0\end{aligned}$$

where N_1^* and the implied M_T^* are the solutions.

Dual problem

- Functional equation:

$$\epsilon = \min_{\lambda} \mathbb{E}\left(\exp\left[-\frac{1}{\xi} g(X_1) + \lambda \cdot f(X_1) \right] \left(\frac{e_1}{e_0} \right) \mid \mathfrak{I}_0 \right)$$

  for $e > 0$ and ϵ a positive number
- Implied probability distortion:

$$M_1^* = \left(\frac{1}{\epsilon^*} \right) \exp\left[-\frac{1}{\xi} g(X_1) + \lambda^* \cdot f(X_1) \right] \left(\frac{e_1^*}{e_0^*} \right)$$

  $\log \epsilon^* = -\frac{\mu^*}{\xi}$
- Alter ξ to hit a relative entropy constraint

Unitary risk aversion

- Consider a recursive utility model Impose $\gamma = 1$ and explore belief distortions instead of large and/or time-varying risk aversion.
- Let R^w denote the return on wealth. The SDF under rational expectations is $(R^w)^{-1}$. Under distorted beliefs represented by M,

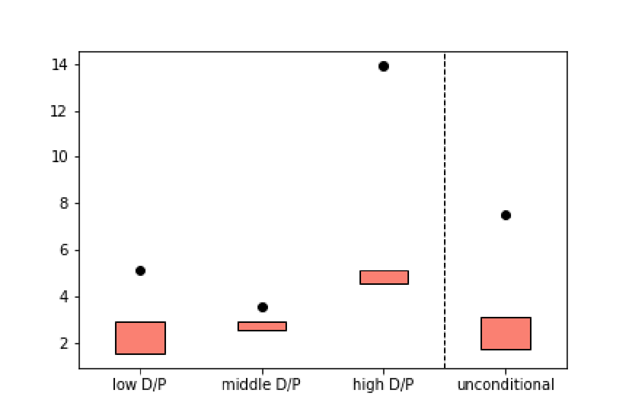
$$S = M(R^w)^{-1}$$

- Consumption Euler equation

$$E\left(M \log R^w \mid \mathfrak{I}\right) = -\log \beta + \rho E\left(M \log G \mid \mathfrak{I}\right).$$

where $\log G$ is the consumption growth rate, $\frac{1}{\rho}$ the elasticity of intermporal substitution, and β the subjective discount factor.

Expected log market return



The ·'s are empirical averages and the boxes give the imputed bounds when we inflated the minimum relative entropy by 20%. The minimum relative entropy is .0284 with a half-life of 24.4 quarters.

Probabilities

Transition matrix $(\cdot)$ and stationary distribution $[\cdot]$

empirical

$$\begin{pmatrix} .96 & .04 & 0 \\ .05 & .88 & .07 \\ 0 & .08 & .92 \end{pmatrix}$$

$$\begin{bmatrix} .42 & .31 & .27 \end{bmatrix}$$

min rel entropy

$$\begin{pmatrix} .98 & .02 & 0 \\ .08 & .88 & .04 \\ 0 & .17 & .83 \end{pmatrix}$$

$$\begin{bmatrix} .76 & .20 & .04 \end{bmatrix}$$

120% min rel entropy, lower bound

$$\begin{pmatrix} .98 & .02 & 0 \\ .10 & .87 & .03 \\ 0 & .19 & .81 \end{pmatrix}$$

$$\begin{bmatrix} .85 & .13 & .02 \end{bmatrix}$$

120% min rel entropy, upper bound

$$\begin{pmatrix} .97 & .03 & 0 \\ .07 & .88 & .04 \\ 0 & .15 & .85 \end{pmatrix}$$

$$\begin{bmatrix} .65 & .27 & .08 \end{bmatrix}$$

Expected market return

conditioning	empirical average	min entropy imputed (lower, upper)	R^w/R^f imputed (lower, upper)
low D/P	6.54 %	3.67 %	3.00
		(2.91, 4.49)	(2.64, 3.37)
mid D/P	4.70 %	3.81 %	2.22
		(3.53, 4.11)	(1.85, 2.68)
high D/P	15.41 %	6.58 %	3.47
		(6.12, 7.05)	(2.88, 4.12)
none	8.94 %	3.82 %	2.86
		(3.07, 4.59)	(2.46, 3.28)

Bounds on logarithm of the expected market return and the logarithm of the expected ratio of the market return to a return on Treasury bills. The numbers in the parentheses impose a relative entropy constraint that is twenty percent higher than the minimum.

Extensions

- While our illustration used the empirical distribution as in GMM type methods, they may also be applied to structural dynamic stochastic equilibrium models in conjunction with VAR type evidence
- We plan to develop further formal econometric methods that support our methods
- Explore implications for policy in an uncertain world