Remarks on Identification, Recovery and Martingales[1]

Recently, Lars was at a very nice event recognizing important contributions of Steve Ross. The "recovery theorem" was discussed multiple times. We write this with a bit of hesitation. We are big fans of Steve Ross' intellectual contributions, and in no way do we seek to undermine his profound impact on financial economics. That said, in spite of the multiple formal analyses of "recovery," there remains some confusion as was evident in the conference discussions. We wrote this note to provide some clarity and push the discussion away from purely technical concerns and back to basic economic considerations. Of course, the technical issues are also key, but apparently the economic implications are not fully appreciated. Consistent with Steve's enthusiasm for research, we aim to provide continued effort and energy to the discussion of research of interest to Steve and to us.

The discussion of "recovery" in finance is a special case of what econometricians call identification. Suppose we have at our disposal (hypothetically) Arrow prices over a one time period in discrete time or over an instant in continuous time. We also impose a Markov structure on the prices. The question is what can be learned about beliefs and discounting. We consider the simplest case of a single stand-in or representative consumer. There are some interesting extensions that look at more general environments and discuss implications for Arrow prices over multiple time periods, but we will not discuss those in what follows.

- *What is the relation between Hansen and Scheinkman (*Econometrica*, 2009), Borovička, Hansen and Scheinkman (*Journal of Finance*, 2016) and Ross (*Journal of Finance*, 2015)?*

  We show how to use versions of Perron-Frobenius theory to extract martingale components to valuation using an "operator approach." We built on a mathematical theory of "multiplicative functionals" and semigroups because they provide a powerful and elegant framework for studying valuation over multiple time horizons. Roughly speaking, but fine for the purposes of this discussion, multiplicative functionals and processes are constructed from an underlying Markov process for which the logarithm of the process has stationary Markov increments. This mathematical structure builds on a convenient Markov representation of asset valuation.

  Our application of Perron-Frobenius methods to valuation is essentially the same as that used by Ross. Perron-Frobenius methods target an "eigenvalue" that is largest

[1] We thank Peter Carr for comments.

in magnitude, strictly positive, and can be shown to dominate over long investment horizons. The underlying goal of our research is to provide valuation methodology that supports the period by period growth due to productive investments or stochastic discounting, which captures impatience and aversion to risk or ambiguity. The Perron-Frobenius theory allows for transient and durable contributions to cash flows to their valuation, while targeting the long-term impacts. Our application of this theory is motivated in part by Alvarez and Jermann (*Econometrica*, 2005) and more generally by uncertainty contributions to valuation that could have permanent consequences and contribute to long-term risk return tradeoffs. Viewing stochastic discount factors as a so-called "multiplicative process" was a natural starting point. Within our framework, the Ross result presumes that the martingale contributions that we sought to characterize were degenerate. The point we make in our paper, "Misspecified Recovery" is that Ross' result is not wrong, but that it is based on some additional restrictions ruled out by many structural models of asset pricing.

- *How does identification work in the finite-state Markov chain example?*

  What follows is essentially lifted from the front end of Borovička, Hansen and Scheinkman. Suppose that a discrete-time Markov process has n states. For each of the n current states, there are n states that could occur in the next time period. Thus, there are n^2 Arrow prices. Stack all of the Arrow prices into an $n \times n$ matrix $\mathcal{A}$ and find the eigenvalue that is largest in magnitude. In discrete time, it will be positive as will be the corresponding eigenvector. Raising the one-period Arrow price matrix $\mathcal{A}$ to an integer power gives the implied prices over the time horizon given by the integer power. Now consider a potential probability matrix that we seek to identify. This matrix can be expressed in terms of $n^2 - n$ free entries as row sums must add up to one. To relate these to the one-period Arrow prices, we have to take account of stochastic discounting, for which we discount the future and adjust for uncertainty. This puts n^2 more free parameters to consider. Thus at this abstract level we have $2n^2 - n$ free parameters (subject to some inequality restrictions) to infer from the n^2 Arrow prices. We are in a situation that an econometrician would refer to as *underidentification*. Of course, we can address this by imposing more restrictions on the stochastic discounting as is commonly done.

  In this Markov chain environment, consistent with both Ross and Hansen and Scheinkman, we use the Perron-Frobenius eigenvalue equation to construct *a* probability measure. Note that the Perron-Frobenius eigenvalue ρ for an eigenvector $\mathbf{e}$ with positive entries satisfies:

$$\mathcal{A}\mathbf{e} = \rho\mathbf{e}.$$

Form:

$$p_{ij} = \frac{a_{ij}e_j}{\rho e_i}$$

where e_i is entry i of the vector $\mathbf{e}$. Fill out the matrix $\mathcal{P} = [p_{ij}]$. This is a valid probability matrix, but what have we identified? Ross takes this as a way to infer investor beliefs, whereas for Hansen and Scheinkman, this is an interesting probability measure that captures long-term risk return tradeoffs. The point is that without further restrictions, we cannot claim that these probabilities are the subjective beliefs of investors. Thus, it is appropriate to focus the discussion on the economic rationale of the additional restrictions.

Not surprisingly the construction of a probability measure in this manner can be extended to more general Markovian environments. I will give another example subsequently.

- *What about long-term risk return tradeoffs?*

There is a very nice earlier paper by Kazemi (*Review of Financial Studies*, 1992) that assumes a stationary Markov process for consumption and discounted, time separable expected utility. Under a power utility, this analysis extends by allowing for a time trend in logarithms. This is an environment in which recovery as envisioned by Ross works as the positive eigenvector informs us of the marginal utilities in each realized state. The eigenvalue captures discounting and possibly trend growth. Kazemi does not use Perron-Frobenius theory explicitly, but he makes an important observation. The reciprocal of the one-period holding period return of a (arbitrarily) long-term discount bond acts as a stochastic discount factor. Arguably, this has strong empirical implications under rational expectations. It can be tested as in Hansen and Singleton (*Econometrica*, 1982) provided that there is a good approximate measurement of this holding period return. The Kazemi finding carries over to the construction we just described of a probability measure using Perron-Frobenius theory. It also gives a formal sense in which long-term risk return tradeoffs are degenerate. See Borovička, Hansen and Scheinkman for an expanded discussion of this and see Bakshi, Chabi-Yo, Gao (*Review of Financial Studies*, in press) for empirical tests.

Alvarez and Jermann (*Econometrica*) take this insight further by observing that more generally the stochastic discount factor process will have an additional martingale component. They and others seek to measure this component using asset market data. By design, the Perron-Frobenius approach in Hansen and Scheinkman allows for this component to be present.

- *What about structural models used in macro asset pricing?*

  Kazemi identifies one important environment in which the Perron-Frobenius approach identifies investor beliefs. His finding is undermined by i) the presence of stochastic growth in consumption, ii) allowing investors to have recursive utility beliefs or iii) allowing investors to be averse to ambiguity. Indeed Alvarez and Jermann first discussed how recursive utility in conjunction with a stationary or trend stationary consumption process induces a martingale component to stochastic discount factors. This result is easiest to see in the case of a unitary elasticity of substitution. Hansen (*Journal of Political Economy*, 2014), among others, describes the martingale component induced by some alternative forms of ambiguity aversion. Hansen and Scheinkman purposefully consider a broader class of Markov valuation models in order to include a rich collection of structural asset pricing models. As a consequence, the probability recovered using Perron-Frobenius theory includes a component that necessarily absorbs long-term risk return tradeoffs but the recovered probability may differ from investor beliefs.

- *How do these issues play out in continuous-time asset pricing models?*

  This answer is specialized for the math finance audience and is not essential to understanding the economic rationale behind recovery. Of course, mathematical formalism and clarity is important and required for a full understanding of the issue.

  Hansen and Scheinkman's original formulation was aimed at continuous-time Markov models. Carr and Wu (*Journal of Derivatives*, 2012), in their discussion of the Ross recovery, look specifically at diffusion models. As they note (as did Hansen and Scheinkman), there are some nontrivial technical conditions that come into play such as discrete spectra and spectral gaps. While important, these do not seem to be at the "heart" of what makes recovery work.

  Think of asset pricing as implying a family of operators that map payoffs that are functions of Markov states into prices that are functions of Markov states, similarly to how we used Arrow prices to form a matrix. Of course, other payoffs are also of interest, but this payoff collection is sufficiently rich to imply stochastic discount factor processes. These processes may then be used to price more general claims.

There is a different operator for each investment horizon. For time-homogeneous Markov processes, these operators depend on the length of the investment horizon. As we make the horizon small there is "derivative" operator that emerges as a so-called *generator*. For diffusions this is a second-order differential operator consistent with the discussion in Carr and Wu.

In what follows, we will be deliberately casual with some regularity conditions. To connect this to diffusions, imagine that we have a Markov diffusion process in some underlying probability. This could be the probability that governs data generation. Now suppose that we represent prices using a stochastic discount factor process whose logarithm has stationary Markov increments, say

$$d \log S_t = \beta_s(X_t)dt + \alpha_s(X_t) \cdot dW_t$$

where W is a possibly multivariate diffusion. We may infer the generator, heuristically, by computing

$$\mathcal{A}f = \lim_{t \downarrow 0} \frac{1}{t} E\left[S_t f(X_t) | X_0 = x\right] = \left(\beta_s + \frac{1}{2}|\alpha_s|^2\right) f + \mathcal{B}f + (\alpha_s)'\sigma' \frac{\partial f}{\partial x}$$

where $\sigma\sigma'$ is the diffusion matrix and $\mathcal{B}$ is the generator of the Markov process.

Here the local risk free rate is $-\beta_s - \frac{1}{2}|\alpha_s|^2$ and the local risk prices for the Brownian increments are contained in the vector $-\alpha_s$. Consider now a positive martingale with a comparable structure:

$$d \log \widetilde{M}_t = -\frac{1}{2}|\tilde{\alpha}_m(X_t)|^2 dt + \tilde{\alpha}_m(X_t) \cdot dW_t$$

We can always form a new probability measure using $\widetilde{M}$ and a new stochastic discount factor $\widetilde{S} = S\widetilde{M}^{-1}$. Notice that $\log \widetilde{S}$ also has stationary Markov increments. Recovery or identification has to tie the hands of a researcher to preclude such a construction. The pair $(\widetilde{S}, \widetilde{M})$ is indistinguishable from $(S, 1)$.

Perron-Frobenius theory gives us a way to do this. Following Hansen and Scheinkman, solve:

$$E\left[S_t e(E_t) | X_0 = x\right] = \exp(\rho t) e(x)$$

for a positive e. (There are often multiple solutions, but Hansen and Scheinkman discuss selection of which is the interesting one.) Note that

$$\exp(\rho t) S_t \frac{e(X_t)}{e(X_0)} = M_t^*$$

is a martingale identified by a Perron-Frobenius eigenvalue problem. The process $\log M^*$ can be represented as suggested with stationary increments in logarithms. Without further restrictions, using this martingale to change measure would produce a probability, but one that does not necessarily govern investor beliefs. Carr and Wu obtain the Ross recovery by imposing Assumption 6 which is equivalent to restricting

$$\log S_t = \phi(X_t) - \phi(X_0) - \exp(\delta t)$$

in our Markov valuation environment. (Carr and Wu work with a process L attributed to Long that is the reciprocal of S). In this case $e = \frac{1}{\phi}$ and $\rho = -\delta$ and $M^* = 1$. More generally, however, using $M^* \neq 1$ to change measure would produce a probability that is a long-term counterpart to a risk-neutral probability.

- In terms of empirical evidence, Alvarez and Jermann and Bakshi and Chabi-Yo (*Journal of Financial Economics*, 2012) compute lower bounds on the martingale component under rational expectations but with incomplete data on Arrow prices. They argue that the martingale contribution is quantitatively important. In addition to the Bakshi, Chabi-Yo and Gao paper we mentioned previously, there is also a financial mathematics literature that models directly the evolution of interest rates or the evolution of stock prices and examines the presence of the martingale component. On the interest rate side there is the Qin, Linetsky and Nie (2016) paper on bond prices that concludes "Thus, transition independence and degeneracy of the martingale component are implausible assumptions in the bond market." Dillschneider and Maurer (2017) use S&P options and also find evidence that Ross recovery is misspecified. We do not doubt that misspecified models are of value (all models are in some sense misspecifed), but it is still valuable to understand what could underlie this misspecification.

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