Stochastic Compounding and Uncertain Valuation*

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Abstract

Exploring long-term implications of valuation leads us to recover and use a distorted probability measure that reflects the long-term implications for risk pricing. This measure is typically distinct from the *physical* and the *risk neutral* measures that are well known in mathematical finance. We apply a generalized version of Perron-Frobenius theory to construct this probability measure and present several applications. We employ Perron-Frobenius methods to i) explore the observational implications of risk adjustments and investor beliefs as reflected in asset market data; ii) catalog alternative forms of misspecification of parametric valuation models; and iii) characterize how long-term components of growth-rate risk impact investor preferences implied by Kreps-Porteus style utility recursions.

1 Introduction

To build structural models of asset valuation and the macroeconomy, empirical researchers use asset market data to learn about investors' preferences, including their beliefs and aver-

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sion to risk. When studying assets with stochastic cash flows, empirical analyses often focus on the implied risk-return tradeoff over short, say one-period, investment horizons. (See van Binsbergen et al. (2012) for a recent exception.) In contrast, we describe methods that feature the interplay between stochastic discounting and growth over alternative horizons. We apply these methods to understand how compounding stochastic growth and discounting alters valuation and thus provide new perspectives on the information content of asset prices. Our characterizations allow for nonlinear stochastic specifications outside the realm of log-normal models or approximations often used in asset pricing models. The methods we describe give characterizations of the components of valuation that become more prominent over longer investment horizons. Specifically, we consider three related substantive problems.

First, we address the question of identifying investor's beliefs and stochastic discounting from asset prices observed at a point in time. To frame this question, we use the mathematically convenient construct of a stochastic discount factor process. Stochastic discount factors represent market valuations of risky cash flows. They are stochastic in order to adjust for risk as they discount the future. Multi-period risk adjustments reflect the impact of compounding single-period risk adjustments. Stochastic discount factors are only well defined relative to a given probability distribution. As is well known, when markets are complete and market prices are observed by an econometrician, for a given probability measure there is a unique stochastic discount factor process that is revealed from financial market data. Ross (2013) extends this claim to argue that the probability measure itself can also be recovered. By studying a class of stochastic economies with growth, we show that the recovery approach of Ross (2013) typically does not recover the underlying probability measure but instead recovers a long-term counterpart to a forward measure. See Section 3

Second, we explore misspecification of parametric models of valuation. It is common to identify stochastic discount factor processes through parametric restrictions. Parametric models, while tractable, are typically misspecified. By studying how the consequences of the misspecification are related to the payoff or investment horizon, we motivate and catalog different forms of misspecification. For instance, economic fundamentals as specified in a parametric model could dominate over longer investment horizons. Thus econometric identification should reflect this possibility. Alternatively, statistically small forms of model misspecification might become more evident over longer time periods if the source is the misspecification of the underlying stochastic evolution as perceived by investors. See Section 4.

Third, recursive specifications of preferences of the type first suggest by Kreps and Porteus

(1978) are known to have nontrivial implications for economies with stochastic growth and volatility. By looking at long-term stochastic characterizations of consumption, we obtain tractable mathematical characterizations of the limiting specifications and in turn general characterizations of when solutions exist for the infinite horizon version of Kreps and Porteus (1978) preferences. In effect, we isolate long-term contributions to the risk-adjustments embedded in recursive utility. For specifications in which the impact of the future in the continuation values is particularly prominent, we show what aspects of the riskiness of the consumption process dominate valuation from the perspective of investor preferences. See Section 5.

We study these three problems using a common analytical approach. This approach starts by characterizing the limiting impact of compounding in a stochastic environment. It applies a generalized version of Perron-Frobenius theory for Markov processes and has much in common with large deviation theory as developed by Donsker and Varadhan (1976). In Section 2 we describe the stochastic environment that underlies our analysis. We show how to use Perron-Frobenius theory to identify a state invariant growth or discount rate and an associated martingale. This martingale induces an alternative probability measure that helps reveal the long-term contributions to growth and valuation. Thus Section 2 lays out the mathematical tools that facilitate our study in subsequent sections.

2   A Factorization Result

Many examples in economics and finance feature a Markov environment and stochastic processes that display stochastic growth. The implied equilibrium stochastic discount processes display stochastic decay as the investment horizon increases. To capture this behavior, the logarithms of the stochastic growth or discount factor processes are modeled conveniently as having stationary increments. In this section, we show how to use *Perron-Frobenius Theory* to decompose such processes as the product of three positive processes: A deterministic trend, a martingale with stationary increments in logarithms and a stationary process. Unless the martingale component is trivial, it makes a stochastic contribution to the growth or discounted embedded in the original process. As is familiar from mathematical finance, there is a change of measure associated with the positive martingale component. When we use the martingale component to change measures, we preserve the Markov property but produce a different transition density for the Markov state. We exploit implications of this factorization in the remainder of the paper.

We begin with a Markov representation of stochastic growth or discounting. This representation is convenient for analyzing the impact of compounding in a stochastic environment. Here we use a discrete-time formulation as in Hansen and Scheinkman (2012a). Continuous-time counterparts have been developed in Hansen and Scheinkman (2009) and Hansen (2012). We apply a general version of Perron-Frobenius theory in making this investigation. Recall that in the case of a matrix A, such that A^k has all strictly positive entries for some positive integer k, Perron-Frobenius theory implies that A has a positive eigenvalue that is associated with a positive eigenvector. This positive eigenvalue dominates in absolute value all other eigenvalues of A and thus dominates the exponential growth rate of A^k as $k \to \infty$. Perron-Frobenius theory generalizes to a class of non-negative linear operators. By applying this approach, we isolate components of growth and valuation that become much more prominent over longer horizons.

We start with a joint Markov process (X, Y):

Assumption 2.1. *The joint Markov process (X, Y) is stationary.*

We could weaken this by imposing some form of stochastic stability, while not initializing the process using the stationary distribution.

Assumption 2.2. *The joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t.*

In light of this restriction, we may view X alone as a Markov process and Y does not "cause" X in the sense of Granger (1969). Moreover, the process Y can be viewed as an independent sequence conditioned on the entire process X where the conditional distribution of Y_t depends only on X_t and X_{t-1}. The role of Y is as a device to introduce an additional source of randomness, but it allows us to focus on the intertemporal impact using a smaller state vector process X.

Construct a process M such that

$$\log M_{t+1} - \log M_t = \kappa(X_{t+1}, Y_{t+1}, X_t).$$

This process has stationary increments. As a result of this construction, the process M will grow or decay stochastically over time, and it is convenient to have methods to characterize this stochastic evolution. Examples of M include stochastic growth processes. These processes could be macro times series expressed in levels that inherit stochastic growth along

some balanced growth path or stochastic discount factor processes used to represent equilibrium asset values. In what follows, we will also have use for

$$\bar{\kappa}(X_{t+1}, X_t) = \log E \left(\exp\left[\kappa(X_{t+1}, Y_{t+1}, X_t) \right] | X_{t+1}, X_t \right),$$

The following example illustrates this construction:

Example 2.3. *Consider a dynamic mixture of normals model. Suppose that X evolves as an n-state Markov chain and that Y is an iid sequence of standard normally distributed random vectors in $\mathbb{R}^m$. Let the realized values of the state vector X_t be the coordinate vectors u_i for $i = 1, 2, ..., n$ where u_i is a vector of zeros except in entry i where it equals 1. Suppose $\bar{\beta} \in \mathbb{R}^n$ and $\bar{\alpha}$ is a $m \times n$ matrix. Let*

$$\log M_{t+1} - \log M_t = \kappa(X_{t+1}, Y_{t+1}, X_t) = \bar{\beta} \cdot X_t + (X_{t+1})' \bar{\alpha}' Y_{t+1}.$$

The current state X_t affects the growth in M in two ways. First, if the state at time t is i, the average growth of (log) M between t and $t+1$ is $\bar{\beta}_i$. In addition, the state at t determines the distribution of X_{t+1} and thus the variance of the growth of (log) M. In a more elaborate model, the state could also affect the distribution of Y_{t+1}, but in this example we assume that Y is iid.

Since $\kappa(X_{t+1}, Y_{t+1}, X_t)$, conditioned on X_t and X_{t+1}, is normally distributed,

$$\bar{\kappa}(X_{t+1}, X_t) = \bar{\beta} \cdot X_t + \frac{1}{2}(X_{t+1})' \bar{\alpha}' \bar{\alpha} X_{t+1}.$$

Next we develop the discrete-time counterpart to an approach from Hansen and Scheinkman (2009) and Hansen (2012). This approach extends a lognormal formulation in Hansen et al. (2008). While log-normal specifications are commonly used because of their convenience they limit the channels by which locally subtle statistical components can become prominent. We use the process M to construct one-period operators and then explore the impact of applying these operators in succession multiple times. This sequential application reflects the impact of compounding. The first operator $\mathbb{M}$ maps functions $g(x, y)$ into functions of the state variable x, $\mathbb{M}g$ via:

$$
\begin{aligned}
[\mathbb{M}g](x) &= E \left(\exp[\kappa(X_{t+1}, Y_{t+1}, X_t)] g(X_{t+1}, Y_{t+1}) | X_t = x \right) \\
&= E \left(\frac{M_t + 1}{M_t} g(X_{t+1}, Y_{t+1}) | X_t = x \right).
\end{aligned}
\tag{1}
$$

Notice that the stationarity of (X, Y) makes the right hand side independent of t. The second operator $\overline{\mathbb{M}}$ maps functions $f(x)$ of the state variable x into functions of the state variable x, $\overline{\mathbb{M}}f$ via:

$$[\overline{\mathbb{M}}f](x) = E\left(\exp[\bar{\kappa}(X_{t+1}, X_t)]f(X_{t+1})|X_t = x\right)$$

The function $g(x, y)$ may be independent of y, that is $g(x, y) = f(x)$. In this case the two operators are consistent. The Law of Iterated Expectations insures that:

$$\begin{aligned}[\mathbb{M}f](x) &= E\left(\exp\left[\kappa(X_{t+1}, Y_{t+1}, X_t)\right] f(X_{t+1})|X_t = x\right) \\ &= E\left(\exp\left[\bar{\kappa}(X_{t+1}, X_t)\right] f(X_{t+1})|X_t = x\right) \\ &= [\overline{\mathbb{M}}f](x)\end{aligned}$$

Thus $\overline{\mathbb{M}}$ and $\mathbb{M}$ agree when they are both well defined. For now we will be vague about the collection of functions g or f that are in the domain of the operators $\mathbb{M}$ and $\overline{\mathbb{M}}$.

Since $\mathbb{M}g$ is only a function of the state x, for any $j \geq 1$,

$$[\mathbb{M}^j g](x) = \overline{\mathbb{M}}^{j-1}[\mathbb{M}g](x).$$

This relation shows that, when we look across multiple horizons, we can concentrate our attention on the operator $\overline{\mathbb{M}}$ and its iterates featuring the X dependence.

In addition, again applying the Law to Iterated Expectations we obtain:

$$[\mathbb{M}^j g](x) = E\left[\left(\frac{M_{t+j}}{M_t}\right) g(X_{t+j}, Y_{t+j})|X_t = x\right].$$

Finally, since M is a positive process, the operators $\mathbb{M}$ and $\overline{\mathbb{M}}$ map positive functions into positive functions.

Example 2.4. *It is straightforward to compute the operator $\overline{\mathbb{M}}$ in Example 2.3. Since X is an n-state Markov chain, functions of x can be identified with vectors in $\mathbb{R}^n$, and the linear operator $\overline{\mathbb{M}}$ can be identified with its matrix representation $A = [a_{ij}]$. Applying $\overline{\mathbb{M}}$ to u_j amounts to computing Au_j and reveals the j-th column of A. Let $P = [p_{ij}]$ be the transition matrix for X. Then it is easy to show that:*

$$a_{ij} = p_{ij}\xi_{ij} \tag{2}$$

where

$$\xi_{ij} \doteq \exp\left[\bar{\beta} \cdot \mathsf{u}_i + \frac{1}{2}(\mathsf{u}_j)'\bar{\alpha}'\bar{\alpha}\mathsf{u}_j\right].$$

Thus the probabilities are adjusted by growth or decay factors, ξ_{ij}, but all entries remain positive. If A^{j_0} has strictly positive entries for some j_0, then the Perron-Frobenius Theorem states that there exists a unique (up to scale) eigenvector e satisfying:

$$A\mathsf{e} = \exp(\eta)\mathsf{e}$$

with strictly positive entries. The eigenvalue associated with this positive eigenvector is positive and has the largest magnitude among all of the eigenvalues. As a consequence, this eigenvalue and associated eigenvector dominate as we apply $\overline{\mathbb{M}}$ j times, for j large.

For general state spaces we consider the analogous Perron-Frobenius problem, namely, find a stricly positive function $e(x)$ such that there exists an η with

$$[\mathbb{M}e](x) = [\overline{\mathbb{M}}e](x) = \exp(\eta)e(x)$$

Existence and uniqueness are more complicated in the case of general state spaces. Hansen and Scheinkman (2009) presents sufficient conditions for the existence of a solution, but even in examples commonly used in applied work, multiple (scaled) positive solutions are a possibility. See Hansen and Scheinkman (2009) and Hansen (2012) for such examples. However, when we have a solution of the Perron-Frobenius problem, we follow Hansen and Scheinkman (2009), and define a process $\widetilde{M}$ that satisfies:

$$\frac{\widetilde{M}_t}{\widetilde{M}_0} = \exp(-\eta t)\left(\frac{M_t}{M_0}\right)\frac{e(X_t)}{e(X_0)}. \tag{3}$$

While this leaves open the question of how to initialize $\widetilde{M}_0$, the initialization is inconsequential to much of what follows as long as $\widetilde{M}_0$ is strictly positive. Notice that by construction:

$$E\left(\frac{\widetilde{M}_{t+1}}{\widetilde{M}_t}|\mathcal{F}_t\right) = E\left(\frac{\widetilde{M}_{t+1}}{\widetilde{M}_t}|X_t\right) = 1$$

implying that $\widetilde{M}$ is a positive $\{\mathcal{F}_t : t = 0, 1, ...\}$-martingale, when $\mathcal{F}_t$ is the sigma algebra of

events generated by $X_0, X_1, ..., X_t$ and $Y_1, Y_2, ..., Y_t$. Moreover,

$$\log \widetilde{M}_{t+1} - \log \widetilde{M}_t = -\eta + \kappa(X_{t+1}, Y_{t+1}, X_t) + \log e(X_{t+1}) - \log e(X_t)$$
$$\doteq \tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t).$$

Thus the process $\widetilde{M}$ has the same mathematical structure and the original process M. In logarithms, this constructed process has stationary increments represented as a function of X_{t+1}, Y_{t+1} and X_t.

An outcome of this construction is the factorization that helps us to understand how compounding works in this Markov environment. Inverting (3):

$$\frac{M_t}{M_0} = \exp(\eta t) \left(\frac{\widetilde{M}_t}{\widetilde{M}_0} \right) \left[\frac{\tilde{e}(X_t)}{\tilde{e}(X_0)} \right] \tag{4}$$

where $\tilde{e} = \frac{1}{e}$. The first term in this factorization determines a deterministic growth (when $\eta > 0$) or decay (when $\eta < 0$) of the M process. The last term is a fixed function of the (stationary) process X_t. Except when the martingale is degenerate, the middle term is a stochastic contribution to compounding. The positive martingale term can be used for a "change of measure". Specifically,

$$[\widetilde{\mathbb{M}}f](x) = E\left(\exp\left[\tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t) \right] f(X_{t+1}) | X_t = x \right)$$

defines the conditional expectation operator implied by the transition distribution for a Markov process. The term $\exp\left[\tilde{\kappa}(X_{t+1}, Y_{t+1}, X_t) \right]$ is the relative density or Radon-Nykodym derivative or a new transition distribution relative to the original transition distribution. In what follows we suppose that the following stochastic stability condition is satisfied

$$\lim_{j \to \infty} [\widetilde{\mathbb{M}}^j f](x) = \int f(x) \widetilde{Q}(dx) \tag{5}$$

where $\widetilde{Q}$ is a stationary distribution for the transformed Markov process. When there are distinct solutions to the Perron-Frobenius problem, at most one solution will satisfy this stochastic stability requirement (see Hansen and Scheinkman (2009)). This stochastic stability is what permits the change of probability measure to be valuable for characterizing long-horizon limits. The change-of-measure captures the long-term impact of the stochastic component to compounding.

In fact, we may write

$$\exp(-\eta j)[\mathbb{M}^j f](x) = e(x)E\left[\frac{\widetilde{M_j}}{\widetilde{M_0}}f(X_j)\tilde{e}(X_j)|X_0 = x\right] = e(x)[\widetilde{\mathbb{M}}^j(f\tilde{e})](x)$$

The stability condition (5) allows us to interpret η as the asymptotic rate of growth. We next explore some applications of this factorization.

3   Stochastic discount factors

In dynamic economic models, stochastic discount factors are used to map future payoffs into current prices. As in Hansen and Richard (1987), we use stochastic discount factors both to discount the future and adjust for risk. They serve as "kernels" for pricing operators that assign current period prices to future payoffs. Standard price-theoretic reasoning connects these discount factors to intertemporal marginal rates of substitution of investors. We presume that the stochastic discount factor S has the mathematical structure described in the previous section for a generic process M, and we use the process S to construct a family of valuation operators indexed by the investment horizon. Recall that the positive martingale constructed in Section 2 defines a probability measure. We show how this alternative probability measure can be revealed by financial market data. When the martingale that induces the alternative probability measure is a constant, the alternative measure is identical to the actual probability measure, and thus we obtain the "recovery" result of Ross (2013). In an equilibrium model, however, the martingale of interest is unlikely to be a constant unless utility is time-separable and consumption is stationary (up to a deterministic rate of growth) as in examples discussed in Ross (2013).[1] In the more general stochastic environment used in this paper, the probability measure recovered using Perron-Frobenius is typically not the one-period transition distribution, but it is an altered measure that can be used to characterize value implications for long investment horizons.

Let $\mathbb{S}$ be the valuation operator that corresponds to the stochastic discount factor, assigning date t prices to date $t+1$ payoffs that are functions of a Markov state, as in equation (1).[2]

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[1]See also Alvarez and Jermann (2005) for a discussion of the empirical importance for a martingale component to stochastic discount factors.

[2]Thus we consider payoffs that are functions of both X_{t+1} and Y_{t+1}. To infer the prices of multi-period payoffs we iterate on the one-period valuation operator. Given a characterization of the pricing of these

We apply factorization (4) to the stochastic factor process:[3]

$$\frac{S_t}{S_0} = \exp(\eta t) \left(\frac{\widetilde{S}_t}{\widetilde{S}_0} \right) \left[\frac{\tilde{e}(X_t)}{\tilde{e}(X_0)} \right]. \tag{6}$$

where $\tilde{e} = \frac{1}{e}$. and e, η are solutions to the Perron-Frobenius problem associated with $\mathbb{S}$

$$\mathbb{S}e = \overline{\mathbb{S}}e = \exp(\eta)e.$$

As before, the operator $\overline{\mathbb{S}}$ is the restriction of $\mathbb{S}$ to payoffs that are only a function of the state variable x. Since S discounts, we expect η to be negative. Additionally, $-\eta$ is the limiting interest rate on a long-term discount bond provided that $\tilde{e}$ has a finite expectation under the change of measure induced by the positive martingale $\widetilde{S}$.

Alvarez and Jermann (2005) use factorization (6) to argue for the importance of permanent shocks as operating through the martingale component $\widetilde{S}$. Specifically, they interpret the multiplicative factorization analogously to an additive counterpart obtained, say by taking logarithms of S. The additive martingale extraction is familiar from time series analysis and empirical macroeconomics as a device to identify permanent shocks.[4] The martingale component in a multiplicative factorization is positive and has "unusual" sample path properties. It converges almost surely, and for many example economies with stochastic growth it converges to zero.[5] Hansen and Scheinkman (2009) and Hansen (2012) instead use the martingale component as a change of measure and show in what sense this change of measure has permanent consequences for pricing. As emphasized by Hansen (2012), if $\log S$ has a nondegenerate martingale component then so does S, and conversely. This relation gives a different but less direct way to motivate the analysis of permanent shocks in Alvarez and Jermann (2005). While there is a tight connection between the multiplicative martingale component of S and the additive martingale component of $\log S$ for log-normal specifica-

"primitive" payoffs, we can extend the valuation operator to an even richer collection of asset payoffs with more complicated forms of history dependence.

[3]See Backus et al. (1989) for an early use of Perron-Frobenius theory in their study of the term structure implications.

[4]The "left over" part in additive decomposition is correlated with the permanent part, making it hard to use directly as a statistical decomposition. This correlation has led many researchers to identify "transitory shocks" as those that are uncorrelated with the martingale increment. Similarly, the Peron-Frobenius eigenfunction is correlated with with the martingale component, making it difficult to interpret (6) directly as a decomposition.

[5]In contrast, the martingale from an additive extraction obeys a central limit theorem when appropriately scaled.

tions, in general there is no simple relation. In what follows we discuss further the implied change in probability distribution associated with the martingale component.

As we argued in Section 2, we may use $\widetilde{S}$ to define a distorted conditional expectations operator as featured in Hansen and Scheinkman (2009) and Hansen (2012):

$$\widetilde{E}\left[g(X_{t+1}, y_{t+1})|X_t = x\right] \doteq E\left[\left(\frac{\widetilde{S}_{t+1}}{\widetilde{S}_t}\right)g(X_{t+1}, Y_{t+1})|X_t = x\right]$$

$$= \exp(-\eta)\tilde{e}(x)[\mathbb{S}(ge)](x) \tag{7}$$

where as before, $\tilde{e} = \frac{1}{e}$. When markets are (dynamically) complete there is a unique operator $\mathbb{S}$ compatible with the observed asset-prices and the absence of arbitrage. If a researcher has at his disposal the relevant information on asset prices, he can infer the operator $\mathbb{S}$, and the (relevant) associated Perron-Frebenius eigenfunction e and eigenvalue $\exp(\eta)$. Alternatively, he might have a more limited amount of asset market data and instead infer $\mathbb{S}$ by parameterizing an underling economic model. Thus right-hand side of (7) sometimes can be identified in a formal econometric sense, revealing the distorted expectation operator on the left-hand side. The recovered transition distribution is *not* the actual one-period transition distribution. Instead it is an altered measure that provides a convenient way to characterize value implications for long investment horizons. It can be viewed as the limiting analog of a forward measure used sometimes in mathematical finance.

Example 3.1. *It is revealing to consider the special case of a finite-state Markov chain for X. For the time being, we abstract from the role of Y and consequently restrict $\mathbb{S} = \overline{\mathbb{S}}$. The conditional expectation for such a process can be represented as a $n \times n$ matrix of transition probabilities, $P = [p_{ij}]$, and functions of the Markov state can be represented as vectors f where entry i is the value that the function takes in state i. Thus as in Example 2.3 there is also a matrix depiction $A = [a_{ij}]$ of the operator $\overline{\mathbb{S}}$ with n^2 entries. and applying $\overline{\mathbb{S}}$ to the ith coordinate vector (a vector with all zeros except in entry i) reveals the ith column of the matrix used to represent the operator $\overline{\mathbb{S}}$. In matrix terms, we solve*

$$A\mathsf{e} = \exp(\eta)\mathsf{e}.$$

It is of interest to consider the "inverse" of relation (7):

$$[\overline{\mathbb{S}}f](x) = \exp(\eta)\widetilde{E}\left[f(X_{t+1})\tilde{e}(X_{t+1})|X_t = x\right]e(x). \tag{8}$$

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Perron-Frobenius theory suggests representing $\overline{\mathbb{S}}$ *by the expression on the right-hand side of* (8). *Using the matrix representation* A, *equation (8) becomes:*

$$a_{ij} = \exp(\eta)\tilde{p}_{ij}\left(\frac{\mathsf{e}_i}{\mathsf{e}_j}\right) \tag{9}$$

where the distorted transition matrix $\widetilde{P} = [\tilde{p}_{ij}]$. *Restricting the* $\tilde{p}_{ij}$ *to be transition probabilities in (9) guarantees that if* $\widetilde{P}$ *and the vector* e *solve the* n^2 *equations (9) then*

$$(Ae)_i = \sum_j a_{ij}\mathsf{e}_j = \exp(\eta)\sum_j \tilde{p}_{ij}\mathsf{e}_i = \exp(\eta)\mathsf{e}_i,$$

and hence e *is indeed an eigenvector of* A *associated with eigenvalue* $\exp(\eta)$. *The transition matrix,* $\widetilde{P}$ *has* $(n^2 - n)$ *free parameters. The* n *dimensional eigenvector* e *with positive entries is only identified up to scale and hence depends on* $n - 1$ *free parameters and the eigenvalue* $\exp(\eta)$ *gives one more free parameter. Thus, for this example we may think of (8) or equivalently (9) as providing* n^2 *equations with input* A *to be used in identifying* n^2 *free parameters of* $\widetilde{P}$, e *and* η. *While Perron-Frobenius approach recovers a transition matrix, it is not the transition matrix for the underlying Markov chain.*

If $\widetilde{S}$, the martingale component of S, is constant over time, the distorted and actual probabilities are identical and one obtains the recovery result of Ross (2013), who also uses Perron-Frobenius theory in his construction. This is a remarkable result because it allows for the recovery of the transition distribution of the Markov state and thus permits investor beliefs to be subjective. When marginal utility of an investor is a time-invariant function of the Markov state, we may interpret $\delta = -\eta$ as a subjective discount rate and $\tilde{e}$ as the marginal utility of consumption expressed as a function of the Markov state.

To justify the recovery of the actual (as opposed to distorted) transition density, we had to restrict the martingale component to be constant over time. Alvarez and Jermann (2005) argue why it is important empirically to allow for the martingale component in a stochastic discount factorization (6). In an equilibrium model, a specification where the martingale component is constant is unlikely to hold unless consumption is stationary, an assumption that typically is not made in the macro-asset pricing literature.[6] In general, the application of Perron-Frobenius theory to one-period valuation recovers a transition distribution, but one that characterizes long-term valuation. This transition distribution will differ from

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[6]As Alvarez and Jermann (2005) point out, even if consumption is stationary, the more general recursive utility model could imply a martingale component to the stochastic discount factor process.

the actual conditional distribution when there is a martingale component to the stochastic discount factor.

Example 3.2. *Consider the dynamic normal mixture model from Example 2.3. In that example it is assumed that*

$$\log S_{t+1} - \log S_t = \bar{\beta} \cdot X_t + (X_{t+1})' \bar{\alpha}' Y_{t+1}$$

This dynamics for S can be justified from more primitive assumptions. For instance, if the representative consumer has log *utility and a subjective discount rate δ, the dynamics for S follow, provided we postulate that the consumption process C satisfies:*

$$\log C_{t+1} - \log C_t = \tilde{\beta} \cdot X_t + (X_{t+1})' \tilde{\alpha}' Y_{t+1} \tag{10}$$

where

$$\bar{\beta} = -\tilde{\beta} - \exp(-\delta)\mathbf{1}_n$$
$$\bar{\alpha} = -\tilde{\alpha}$$

and $\mathbf{1}_n$ is an n-dimensional vector of ones. In equation (10), the expected change in $\log C$ between t and t + 1 is given by $\tilde{\beta} \cdot \mathbf{u}_i$ (the ith component of $\tilde{\beta}$) when state i is realized at date t. The variance of the change in $\log C$ is $(\mathbf{u}_j)' \tilde{\alpha}' \tilde{\alpha} \mathbf{u}_j$ (the jth diagonal entry of $\tilde{\alpha}' \tilde{\alpha}$) conditioned on state j being realized at date t + 1. The vector $\bar{\beta}$ and matrix $\bar{\alpha}$ have analogous interpretations for the process $\log S$.

We use this example to better illustrate the link between P and $\widetilde{P}$. We presume that state vector (X, Y) is observable to investors and consider the recovery of the transition matrix for the state vector x. For this example, equations (2) and (9) show that the recovered transition matrix implied by (7) is $\widetilde{P}$ where:

$$\tilde{p}_{ij} = \exp(-\eta) p_{ij} \xi_{ij} \left(\frac{\mathsf{e}_j}{\mathsf{e}_i} \right).$$

Direct verification shows that $\widetilde{P}$ is indeed a transition matrix. This transition matrix agrees with P when

$$\exp(-\eta) \xi_{ij} \left(\frac{\mathsf{e}_j}{\mathsf{e}_i} \right) = 1 \tag{11}$$

for all (i,j), but this will not be true in general. Equation (11) implies that

$$\xi_{ij} = \exp(\eta) \left(\frac{\mathsf{e}_i}{\mathsf{e}_j} \right). \tag{12}$$

The ξ_{ij}'s depend on $2n$ parameters, the n means of $\log S_{t+1} - \log S_t$ and the n variances of $\log S_{t+1} - \log S_t$ conditioned on X_t and X_{t+1}. Thus there are $2n$ parameters that underly the ξ_{ij}'s on the left-hand side of this equation, but only n free parameters that we can vary on the right-hand side (where e only matters up to a free scale parameter).[7]

To make more transparent how restrictive is the requirement in equation (12), suppose that $n = 2$ and Y is univariate. Then

$$\xi_{ij} = \exp \left[\bar{\beta}_i + \frac{1}{2} (\bar{\alpha}_j)^2 \right].$$

If $p_{ii} = \tilde{p}_{ii}$, then $\xi_{ii} = \exp(\eta)$ and thus $\bar{\beta}_1 + \frac{1}{2}(\bar{\alpha}_1)^2 = \bar{\beta}_2 + \frac{1}{2}(\bar{\alpha}_2)^2$.

Appendix A describes an alternative recovery strategy, but also argues why this approach will not work under more general circumstances.[8]

A reader might be concerned that our focus on the finite state case is too special. What follows is an example based on an underlying Markov diffusion specification except that we allow for more shocks to be priced than the underlying state variable.

Example 3.3. *Suppose that X is a Markov diffusion:*

$$dX_t = \mu(X_t)dt + \Lambda(X_t)dW_t$$

and

$$d \log S_t = \beta(X_t)dt + \alpha(X_t) \cdot dW_t.$$

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[7]In this example the ξ_{ij}'s depend on $2n$ parameters but one can construct examples in which there are n^2 free parameters.

[8]One might argue that we should expand our search for eigenfunctions to a broader set of functions. For instance, we might add S_t to the state vector along with X_t and Y_t even though $\log S_{t+1} - \log S_t$ is constructed from X_{t+1}, Y_{t+1} and X_t. Since S is typically not stationary, this would require relaxing some our maintained assumptions and would require that we dispense with the selection rule in Hansen and Scheinkman (2009). A potential gain to the approach is that we might view $\frac{1}{S_t}$ as a Perron-Frobenius eigenfunction associated with the unit eigenvalue based on the observation that $\frac{1}{S_t}$ is strictly positive. One could modify the selection rule of Hansen and Scheinkman (2009) requiring only the stochastic stability of X. This, however, would allow potentially for both candidate solutions. Alternatively, we could identify the eigenfunction of interest by "zeroing out" the dependence on X_t and Y_t. If this is our selection criterion, we would choose $\frac{1}{S_t}$ but in this case the use of Perron-Frobenius theory adds no information concerning the dynamics of X.

The counterpart to Y_{t+1} in this example is any component of the dW_t increment that is not needed for the evolution of X. To construct the counterpart to $\overline{\mathbb{S}}$ in continuous time we consider the operator $\mathbb{G}$ that associates to each function f the "expected time derivative" of $\frac{S_t}{S_0}f(X_t)$ at $t = 0$, that is the drift of $\frac{S_t}{S_0}f(X_t)$ at $t = 0$.[9]

$$[\mathbb{G}f](x) = \left[\beta(x) + \frac{1}{2}|\alpha(x)|^2\right]f(x) + [\mu(X_t)' + \alpha(x)'\Lambda(x)]\frac{\partial f(x)}{\partial x}$$
$$+ \frac{1}{2}\text{trace}\left[\Lambda(x)\Lambda(x)'\frac{\partial^2 f(x)}{\partial x \partial x'}\right]$$

In this expression, $-\beta(x) - \frac{1}{2}|\alpha(x)|^2$ is the instantaneous interest rate and $-\alpha$ is the vector of risk prices that capture the expected return reward needed as compensation for exposure to dW_t. To apply Ito's formula to compute the drift, we may restrict f to be twice continuously differentiable; but we typically have to explore a larger collection of functions in order to solve eigenvalue problem:

$$[\mathbb{G}e](x) = \eta e(x) \tag{13}$$

for a positive function e. See Hansen and Scheinkman (2009) for a more formal analysis.

The diffusion dynamics implied by the Perron-Frobenius adjustment is:

$$dX_t = \left[\mu(X_t)' + \alpha(x)'\Lambda(x)' + \left[\frac{\partial \log e(x)}{\partial x}\right]'\Lambda(x)\Lambda(x)'\right]dt + \Lambda(X_t)d\widetilde{W}_t$$

where $\widetilde{W}$ is a Brownian motion under the change of measure. See Appendix B. The version of equation (7) for continuous time allows us to identify the distorted dynamics for X. Thus in order for this procedure to identify the original dynamics it is necessary that

$$\alpha(x) = -\Lambda(x)'\left[\frac{\partial \log e(x)}{\partial x}\right].$$

As is shown in Appendix B, when this restriction is satisfied we may write:

$$\frac{S_t}{S_0} = \exp(\eta t)\frac{e(X_0)}{e(X_t)},$$

implying that S has a constant martingale component.

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[9]This operator is called the "generator for the pricing semigroup" in Hansen and Scheinkman (2009).

4 Model misspecification

In this section we suggest a novel approach to confronting potential model misspecification. We feature transient model misspecifications whereby the candidate economic model of a stochastic discount factor is restricted to give the correct limiting risk prices and approximately correct processes for long investment payoff horizons. This approach restricts the misspecified model to share the eigenvalue and the martingale component with the true or correctly specified model. We show how to use Perron-Frobenius and cumulative returns to identify the unknown parameters needed to characterize the benchmark misspecified model. We also contrast this approach to models of belief distortions that by design target the martingale components of stochastic discount factors.

As the examples in section 3 suggest, Perron-Frobenius theory recovers an *interesting* probability distribution but typically one that is distorted. To proceed under stochastic growth, we must bring to bear additional information. Otherwise we would be left with a rather substantial identification problem. Following the literature on rational expectations econometrics, we can appeal to "cross equation restrictions" if the Markov state vector process X is observable. In what follows we sketch two related approaches for evaluating parametric stochastic discount factor models of valuation. These approaches are distinct but complementary to the nonparameteric approach proposed by Ait-Sahalia and Lo (2000).[10]

4.1 Transient model misspecification

Building from Bansal and Lehmann (1997) and subsequent research, in Hansen (2012) we relate factorization (4) to a stochastic discount factor representation

$$\frac{S_t}{S_0} = \frac{S_t^*}{S_0^*} \left[\frac{h(X_t)}{h(X_0)} \right] \tag{14}$$

where $\frac{S_t^*}{S_0^*}$ is the stochastic discount factor in representative consumer power utility model and h modifies the investor preferences to include possibly internal habit persistence, external habit persistence or limiting versions of recursive utility. In the case of internal and external habit persistence models, these modifications may entail an endogenous state variable constructed on the basis of current and past consumptions. Introducing this function

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[10]Ait-Sahalia and Lo (2000) show how to recover a stochastic discount factor nonparametrically over a given investment horizon using option prices to extract a forward measure and forming the ratio of the forward transition density to the actual transition density measured from the observed data.

h modifies the Perron-Frobenius problem by leaving the eigenvalue intact and altering the eigenfunction as follows. If e^* is the eigenfunction associated with S^* then $e = e^*/h$ is the Perron-Frobenius eigenfunction for S. See Chen and Ludvigson (2009) and Chen et al. (2011) for semiparametric implementations of this factorization for some specific examples.

Let us turn to the specific application of Chen et al. (2011), but adopt a discrete-time counterpart to a formulation in Hansen and Scheinkman (2009). Let R be a cumulative return process and C a consumption process. Both are modeled with the "multiplicative" structure described in section 2. Form:

$$\frac{M_t}{M_0} = \frac{R_t}{R_0} \left(\frac{C_t}{C_0}\right)^{-\gamma}$$

where γ is the risk aversion parameter for a power utility model of investor preferences. Chen et al. (2011) presume that the actual stochastic discount factor is:

$$\frac{S_t}{S_0} = \exp(-\delta t) \left(\frac{C_t}{C_0}\right)^{-\gamma} \left[\frac{h(X_t)}{h(X_0)}\right]$$

where X is the growth rate in consumption and any other variables that might forecast that growth rate. They motivate h as arising from a consumption externality in which lagged consumption is viewed as being socially determined as in Abel (1990). In contrast to Abel (1990), Chen et al. (2011) aim to be nonparametric, and h becomes a multiplicative adjustment to a marginal utility of consumption that they seek to identify with limited restrictions. Thus they presume that the stochastic dynamics can be inferred and allow for growth in consumption.

A fundamental result in asset pricing is that RS is always a martingale, and as consequence,

$$\frac{M_t}{M_0} = \exp(\delta t) \left(\frac{\widetilde{M_t}}{\widetilde{M_0}}\right) \left[\frac{h(X_0)}{h(X_t)}\right].$$

Thus $\exp(\delta)$ is the associated Perron-Frobenius eigenvalue and h is the eigenfunction. In contrast to Ross (2013), the actual transition dynamics are presumed to be directly identifiable. Perhaps surprisingly h and δ can both be inferred from a single return process provided that we can infer the underlying conditional distributions from historical data.[11] The use of mul-

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[11] A researcher may actually prefer to place some restrictions on h consistent with an underlying preference interpretation. Even so, the identification result in Chen et al. (2011) remains interesting as additional restrictions should only make identification easier.

tiple returns allows for the identification of γ along with over-identifying restrictions. The same eigenvalue and eigenfunction should be extracted when we alter the return process.[12]

We suggest an alternative approach based on a similar idea. Let S^* be a benchmark economic model that is possibly misspecified. We no longer limit our specification S^* to be the stochastic discount factor associated with power utility. Instead we allow for a more general starting point. We restrict the potential misspecification to have transient implications for valuation with the actual stochastic discount factor representable as (14). The function h is introduced to capture transient sources of misspecification. We call this modification transient because the modified stochastic discount factor shares the same eigenvalue η and hence the same long-term interest rate, and it shares the same martingale component, which determines the long-term risk prices as argued in Hansen and Scheinkman (2009), Borovička et al. (2011), Hansen and Scheinkman (2012b), and Hansen (2012). The presumption is that we have little *a priori* structure to impose on h other than limiting its long-term consequences.

Let

$$M = RS^*$$

and recall that $RS = \tilde{M}$ is itself a martingale. Thus we write

$$\frac{M_t}{M_0} = \left(\frac{\widetilde{M_t}}{\widetilde{M_0}} \right) \left[\frac{h(X_0)}{h(X_t)} \right].$$

The Perron-Frobenius eigenvalue $\exp(\eta)$ is now one and the eigenfunction $\mathsf{e} = h$. This representation using the unit eigenvalue and the eigenfunction h must hold whenever R is a cumulative return process for a feasible self-financing strategy. This link across returns allows for the identification of unknown parameters needed to characterize the benchmark model S^* under misspecification.

This is a rather different approach to misspecification from that suggested by Hansen and Jagannathan (1997). The approach we choose here explicitly restricts the misspecification to have transient consequences for valuation. The true S and the modeled S^* must share the same asymptotic decay rate (long-term interest rate) and the same martingale component. This approach does not restrict the magnitude of h in any particular way. This leaves open the possibility that while h has "transient" consequences for pricing, these consequences might well be very important for shorter time investment horizons. Thus in practice one

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[12]In external habit models such as the one in Campbell and Cochrane (1999), the habit stock is a constructed state variable, one for which there is no direct observable counterpart. An additional challenge for identification is to infer, perhaps with weak restrictions, the law of motion for this endogenous state variable.

would also want to assess impact of this proposed transient adjustment over investment horizons of interest in order to characterize the impact of the economic restrictions and the role for misspecification.

4.2   Distorted beliefs

In the previous subsection, we used a long-term perspective to introduce a structured way for economic models to fit better over longer investment horizons. One motivation for this could be a misspecification of the utility functions used to represent investor preferences, but there may be other reasons to suspect model misspecification. The stochastic discount factor specification, as we have used it so far, presumes a correct specification of the transition probabilities for (X, Y) because the discount factors are only defined relative to a probability distribution. We now consider the impact of incorrectly specifying the stochastic evolution of the state variables, and we allow this to have permanent consequences on valuation. Investors themselves may have "incorrect" beliefs or they may act as if they have distorted beliefs. Motivations for this latter perspective include ambiguity aversion or investor ambitions to be robust to model misspecification. See for instance Hansen and Sargent (2001) and Chen and Epstein (2002).

To capture belief distortion as a form of model misspecification, construct

$$\frac{S_t}{S_0} = \left(\frac{S_t^*}{S_0^*}\right)\left(\frac{N_t}{N_0}\right)$$

where S^* is the modeled stochastic discount factor process and N is a martingale. Both processes are assumed to be of the "multiplicative" form introduced in Section 2. The process S is the pertinent one for pricing assets while the process N is introduced to capture distorted beliefs.

Allowing for arbitrary belief distortions is counterproductive from the perspective of building economic models. While introducing investors with preferences that display ambiguity aversion and concerns about robustness give one way to add some structure to this analysis, this approach still depends on parameters that limit the class of alternative probability models to be considered by an investor. More generally, it is of value to characterize or to limit how big is this source of misspecification from a statistical perspective. Here we consider Chernoff (1952)'s notion of entropy motivated explicitly by the difficulty in statistically discriminating between competing models. Markov counterparts to this approach rely on

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Perron-Frobenius theory; see Newman and Stuck (1979). The Chernoff-style calculations are "large deviation calculations" because mistakes occur when there is an unusual realization of a sequence of observations.

To define Chernoff entropy for the statistical discrimination among Markov processes, we focus on the martingale process N. Since N is a martingale, N^θ is a supermartingale for $0 \le \theta \le 1$. Consider:

$$\epsilon(\theta) = -\lim_{t \to \infty} \frac{1}{t} \log E\left[\left(\frac{N_t}{N_0}\right)^\theta |X_0 = x\right].$$

Notice that $\epsilon(0) = 0$, and since N is a martingale $\epsilon(1) = 0$, but more generally $\epsilon(\theta) \ge 0$ because N^θ is a super martingale. As the asymptotic rate of growth of $\exp(\theta \log N) = N^\theta$, $\exp[-\epsilon(\theta)]$ can be seen as the Perron-Frobenius eigenvalue associated with this process. Chernoff entropy is the asymptotic decay rate for making mistakes in determining the correct model from historical data. It is given by solving:

$$\max_{\theta \in [0,1]} \epsilon(\theta).$$

When the maximized value is close to zero it is difficult to distinguish between the original model and the distorted model captured by N. Anderson et al. (2003) suggest this as a way to assess when investors concerns about model misspecification might be reasonable. More generally, this can used to assess how large the misspecification is from a statistical standpoint.[13] Such calculations give us a way to see if *statistically small* but permanent distortions in probability specifications can have notable consequence for valuation.

5   Recursive utility valuation

Recursive utility of the type initiated by Kreps and Porteus (1978) and Epstein and Zin (1989) represents the valuation of prospective future risky consumption processess through the construction of continuation values. This approach avoids the reduction of intertemporal compound lotteries and thus allows for the intertemporal composition of risk to matter. Bansal and Yaron (2004) use this feature of recursive preferences to argue that even statistically subtle components of growth rate risk can have an important impact on valuation. Hansen and Scheinkman (2012a) establish a link between Perron-Frobenius theory and Kreps and Porteus (1978) style recursive utility. We show that this link provides a new perspec-

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[13]Anderson et al. (2003) also explore a local counterpart to the Chernoff measure.

tive on the valuation impacts of stochastic growth and volatility in consumption as they are compounded over time, and it further clarifies the roles of the parameters used in the representing the utility recursion.

We use the homogeneous-of-degree-one aggregator specified in terms of current period consumption C_t and the continuation value V_t:

$$V_t = \left[(\zeta C_t)^{1-\rho} + \exp(-\delta) \left[\mathcal{R}_t(V_{t+1}) \right]^{1-\rho} \right]^{\frac{1}{1-\rho}} \tag{15}$$

where

$$\mathcal{R}_t \left(V_{t+1} \right) = \left(E \left[(V_{t+1})^{1-\gamma} | \mathcal{F}_t \right] \right)^{\frac{1}{1-\gamma}}$$

adjusts the continuation value V_{t+1} for risk. With these preferences, $\frac{1}{\rho}$ is the elasticity of intertemporal substitution and δ is a subjective discount rate. The parameter ζ does not alter preferences, but gives some additional flexibility that is valuable when taking limits. Next exploit the homogeneity-of-degree one specification of the aggregator (15) to obtain:

$$\left(\frac{V_t}{C_t} \right)^{1-\rho} = \zeta^{1-\rho} + \exp(-\delta) \left[\mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right]^{1-\rho}. \tag{16}$$

In finite horizons using the aggregator requires a terminal condition for the continuation value. In what follows we will consider infinite-horizon limits, leading us to consider fixed point equations. Thus we will explore the construction of the continuation value V_t as a function of $C_t, C_{t+1}, C_{t+2}, \ldots$.

Suppose that the consumption dynamics evolve as:

$$\log C_{t+1} - \log C_t = \kappa(X_{t+1}, Y_{t+1}, X_t).$$

Given the Markov dynamics, we seek a solution:

$$\left(\frac{V_t}{C_t} \right)^{1-\rho} = f(X_t), \quad f \geq 0.$$

Writing

$$\alpha = \frac{1-\gamma}{1-\rho}$$

and for $f \geq 0$,

$$[\mathbb{U}f](x) = \zeta^{1-\rho} + \exp(-\delta) \left(E \left[f(X_{t+1})^{\alpha} \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] | X_t = x \right] \right)^{\frac{1}{\alpha}},$$

we express equation (16) as:

$$\hat{f}(x) = [\mathbb{U}\hat{f}](x). \tag{17}$$

The solution to the fixed point problem (17) is closely related to a Perron-Frobenius eigenvalue equation.[14] To see this relation, consider the mapping:

$$[\mathbb{T}f](x) = E\left[\exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)]f(X_{t+1})|X_t = x\right]. \tag{18}$$

The eigenvalue equation of interest is:

$$[\mathbb{T}e](x) = \exp(\eta)e(x) \tag{19}$$

for $e > 0$. Hansen and Scheinkman (2012a) construct a solution to (17) using the eigenfunction $e(x)$ provided the following parameter restriction

$$\delta > \frac{1-\rho}{1-\gamma}\eta, \tag{20}$$

is satisfied, along with some additional moment restrictions.

Why might there be a connection between the eigenvalue equation (19) and the fixed-point equation (17)? Consider equation (17), but add the following modifications:

i) Set $\zeta^{1-\rho} = 0$;

ii) Raise both sides of the equation to the power α;

iii) Set δ to satisfy inequality (20) with equality.

This modified version of equation (17) is essentially the same as the eigenvalue equation (19).

What do we make of this observation? The positive number ζ is merely a convenient scale factor. Its specific choice does not alter the preferences implied by the recursive utility model. This gives us some flexibility in taking limits. The limit that we must take entails letting the subjective discount rate δ approximate the bound in (20). In effect, this makes the future as important as possible in the utility recursion. Hansen and Scheinkman (2012a) discuss more formally how this limit effectively reduces the infinite horizon value function problem to a Perron-Frobenius eigenvalue problem. For many common model parameterizations, the eigenfunction is the exponential of a quadratic and the eigenvalue equation can be solved

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[14]See Duffie and Lions (1992) for a related application of Perron-Frobenius theory as an input into an existence argument.

in a straightforward manner. This solution then can be used to construct a solution to the infinite horizon utility recursive under the stated parameter restrictions.

In the notation of the previous Sections, the process M_t used in conjunction with the Perron-Frobenius theory is

$$\frac{M_t}{M_0} = \left(\frac{C_t}{C_0}\right)^{1-\gamma}.$$

The martingale component for this process provides a convenient change of measure to use for evaluating the utility recursion as it absorbs the stochastic growth component to consumption pertinent for valuation. This approach thus features the risk aversion parameter γ in the construction of a martingale component relevant for the analysis of investor preferences.

6   Conclusion

In this paper we studied three problems using a common approach. This approach assumes a Markov environment and applies Perron-Frobenius theory to characterize the long-run impact of compounding in a stochastic environment. Given a possibly non-linear stochastic process describing growth or discounting, this approach identifies three multiplicative components: i) a deterministic growth or decay rate rate; ii) a transient component and iii) a martingale component. The latter two components are typically correlated. We use the martingale component to produce a distorted probability measure that helps reveal long-term contribution to valuation induced by stochastic growth or discounting.

First, we applied these methods to investigate how compounding interacts with the assignment of risk prices and to examine the possibility of disentangling risk adjustments from investor beliefs using asset market data. We showed that, in the presence of stochastic growth, application of Perron-Frobenius theory does *not* recover the actual probability used by investors but instead recovers the distorted probability measure. This measure is informative about the implications for risk prices over long payoff and investment horizon.

Second, we used the decomposition to examine misspecification of parametric models of valuation. These models, while tractable, are surely misspecified. Departures from economic fundamentals may be of a temporary nature, and econometric identification should reflect this possibility. This can be accomplished in the framework developed in this paper by assuming that the true stochastic discount factor and the parametric model of the discount factor share the same deterministic growth rate and martingale component, differing only in their respective transient components to valuation. We also discussed how to treat the

impact of assuming that investors have "wrong" beliefs in environments in which we allow for the misspecification to have permanent consequences for valuation.

Third, we discussed how to use Perron-Frobenius theory to obtain existence results for infinite-horizon Kreps-Porteus utility functions, and highlighted the role of the risk-aversion parameter in the construction of the stochastic growth component of consumption that is pertinent to long-run valuation.

The methods we describe here can also be applied to study the valuation of unusual episodes as they emerge over multiple time periods. While the episodes might be disguised in the short run, they could become more prominent over longer horizons. In our analysis here, we applied a generalized version of Perron-Frobenius theory for Markov processes, an approach that has much in common with large deviation theory as developed by Donsker and Varadhan (1976). To better appreciate this connection, see Stutzer (2003) and Hansen and Scheinkman (2012a). An alternative approach analyzes such phenomena with a different type of limiting behavior. In future work, we plan to study the pricing of rare events using a large-deviation theory in continuous time whereby we hold fixed the valuation time interval while progressively reducing the exposure to Brownian motion shocks.

A Mixture of Normals example

Consider next an alternative approach to recovery for Example 3.2. The positive random variable:

$$\frac{\frac{S_{t+1}}{S_t}}{E\left(\frac{S_{t+1}}{S_t}|X_{t+1}, X_t\right)}$$

alters the distribution of Y_{t+1} conditioned on X_{t+1} and X_t in the same manner as described previously. The adjustment induced by:

$$E\left(\frac{S_{t+1}}{S_t}|X_{t+1}, X_t\right)$$

is captured in the A matrix. Consider now a payoff of the form $g_1(Y_{t+1})g_2(X_{t+1}, X_t)$ where g_2 is one when X_t is in state i and X_{t+1} is in state j. We may compute this price by first evaluating the distorted expectation of g_1 conditioned on X_{t+1}, X_t and then multiplying this conditional expectation by a_{ij}. Since the a_{ij}'s are can be inferred by asset market data so can the altered distribution for Y_{t+1} conditioned on X_{t+1}, X_t. Thus $\bar{\alpha}$ is revealed. The terms $\bar{\beta} \cdot \mathsf{u}_i$ and the p_{ij}'s can be solved in terms of the a_{ij}'s and the $\frac{1}{2}(\mathsf{u}_j)'\bar{\alpha}'\bar{\alpha}\mathsf{u}_j$'s. Thus P can be recovered but not by a Perron-Frobenius extraction. In the more general case:

$$\log S_{t+1} - \log S_t = (X_t)'\tilde{\beta}X_{t+1} + (X_{t+1})'\bar{\alpha}'Y_{t+1}$$

where $\tilde{\beta}$ is an $n \times n$ matrix then it is typically not possible to recover P without additional restrictions.

B Markov diffusion example

In this section we give some supporting arguments for the discussion of Example 3.3. Suppose that X is a Markov diffusion:

$$dX_t = \mu(X_t)dt + \Lambda(X_t)dW_t,$$

and

$$d\log S_t = \beta(X_t)dt + \alpha(X_t) \cdot dW_t.$$

To construct a transformed generator we compute the drift of $\exp(-\eta t)f(x_t)\left(\frac{e(X_t)}{e(X_0)}\right)$.

This gives rise to the transformed generator:

$$
\begin{aligned}
[\widetilde{\mathbb{G}}f](x) =& \frac{1}{e(x)}[\mathbb{G}(ef)](x) - \eta f(x) \\
=& [\mathbb{G}f](x) + \frac{f(x)}{e(x)}[\mathbb{G}e](x) - \left[\beta(x) + \frac{1}{2}|\alpha(x)|^2\right]f(x) \\
& + \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x)\Lambda(x)' \left[\frac{\partial f(x)}{\partial x}\right] - \eta f(x) \\
=& [\mathbb{G}f](x) - \left[\beta(x) + \frac{1}{2}|\alpha(x)|^2\right]f(x) + \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x)\Lambda(x)' \left[\frac{\partial f(x)}{\partial x}\right] \\
=& \left[\mu(X_t)' + \alpha(x)'\Lambda(x)' + \left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x)\Lambda(x)'\right] \frac{\partial f(x)}{\partial x} \\
& + \frac{1}{2}\text{trace}\left[\Lambda(x)\Lambda(x)'\frac{\partial^2 e(x)}{\partial x \partial x'}\right].
\end{aligned}
$$

As expected, the coefficient on the level $f(x)$ is zero, because we want this to be the generator of a stochastically stable Markov process. The matrix of coefficients for the second derivative of f remains the same as the absolute continuity between the implied probability measures over finite time intervals requires that the diffusion matrix remain the same. The important change is in the vector of coefficients for the first derivative of f. This modification alters the drift of the diffusion in the manner state in the discussion of Example 3.3.

This approach recovers the original generator of the Markov process when e solves the eigenfunction equation (13) and:

$$
\alpha(x) = -\Lambda(x)'\left[\frac{\partial \log e(x)}{\partial x}\right].
$$

Given these restrictions, it follows that

$$
\begin{aligned}
\eta =& \beta(x) - \frac{1}{2}\left[\frac{\partial \log e(x)}{\partial x}\right]' \Lambda(x)\Lambda(x)' \left[\frac{\partial \log e(x)}{\partial x}\right] \\
& + \mu(x)'\frac{\partial \log e(x)}{\partial x} + \frac{1}{2e(x)}\text{trace}\left[\Lambda(x)\Lambda(x)'\frac{\partial^2 e(x)}{\partial x \partial x'}\right] \\
=& \beta(x) + \mu(x)'\frac{\partial \log e(x)}{\partial x} + \frac{1}{2}\text{trace}\left[\Lambda(x)\Lambda(x)'\frac{\partial^2 \log e(x)}{\partial x \partial x'}\right].
\end{aligned}
$$

As a consequence, $\beta(x) = \eta - \text{drift} \log e(x)$ where

$$\text{drift} \log e(x) = \mu(x)' \frac{\partial \log e(x)}{\partial x} + \frac{1}{2}\text{trace}\left[\Lambda(x)\Lambda(x)' \frac{\partial^2 \log e(x)}{\partial x \partial x'}\right]$$

and hence

$$d \log S_t = \eta dt - d \log e(X_t)dt.$$

Integrating between dates zero and t and exponentiating gives:

$$\frac{S_t}{S_0} = \exp(\eta t)\left[\frac{e(X_0)}{e(X_t)}\right].$$

References

Abel, Andrew B. 1990. Asset Prices under Habit Formation and Catching up with the Jones. *merican Economic Review* 80 (2):38–42.

Ait-Sahalia, Yacine and Andrew W. Lo. 2000. Nonparametric Risk Management and Implied Risk Aversion. *Journal of Econometrics* 94:9–51.

Alvarez, Fernando and Urban J. Jermann. 2005. Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth. *Econometrica* 73 (6):1977–2016.

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent. 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection. *Journal of the European Economic Association* 1 (1):68–123.

Backus, David K., Allan W. Gregory, and Stanley E. Zin. 1989. Risk Premiums in the Term Structure : Evidence from Artificial Economies. *Journal of Monetary Economics* 24 (3):371–399.

Bansal, Ravi and Bruce N. Lehmann. 1997. Growth-optimal Portfolio Restrictions on Asset Pricing Models. *Macroeconomic Dynamics* 1 (2):333–354.

Bansal, Ravi and Amir Yaron. 2004. Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance* 59 (4):1481–1509.

van Binsbergen, Jules, Michael Brandt, and Ralph Koijen. 2012. On the Timing and Pricing of Dividends. *American Economic Review* 102 (4):1596–1618.

Borovička, Jaroslav, Lars Peter Hansen, Mark Hendricks, and Joé A. Scheinkman. 2011. Risk Price Dynamics. *Journal of Financial Econometrics* 9:3–65.

Campbell, John Y. and John H. Cochrane. 1999. By Force of Habit. *Journal of Political Economy* 107 (2):205–251.

Chen, Xiahong and Sydney Ludvigson. 2009. Land of Addicts? An Empirical Investigation of Habit-Based Asset Pricing Models. *Journal of Applied Econometrics* 24:1057–1093.

Chen, Xiaohong, Victor Chernozhukov, Sokbae Lee, and Whitney Newey. 2011. Local Identification of Nonparametric and Semiparametric Models. Cowles Foundation Discussion Paper No. 1795, Yale University.

Chen, Zengjing and Larry Epstein. 2002. Ambiguity, Risk, and Asset Returns in Continuous Time. *Econometrica* 70 (4):1403–1443.

Chernoff, Herman. 1952. A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the Sum of Observations. *Annals of Mathematical Statistics* 23 (4):pp. 493–507.

Donsker, Monroe E. and S. R. Srinivasa Varadhan. 1976. On the Principal Eigenvalue of Second-Order Elliptic Differential Equations. *Communications in Pure and Applied Mathematics* 29:595–621.

Duffie, Darrel and Pierre-Louis Lions. 1992. PDE solutions of stochastic differential utility. *Journal of Mathematical Economics* 21 (6):577–606.

Epstein, L. and S. Zin. 1989. Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57 (4):937–969.

Granger, C. W. J. 1969. Investigating Causal Relations by Econometric Models with Cross-spectral Methods. *Econometrica* 37:424–438.

Hansen, Lars Peter. 2012. Dynamic Valuation Decomposition within Stochastic Economies. *Econometrica* 911-967 (3). Fisher-Schultz Lecture at the European Meetings of the Econometric Society.

Hansen, Lars Peter and Ravi Jagannathan. 1997. Assessing Specification Errors in Stochastic Discount Factor Models. *Journal of Finance* 52 (2):557–590.

Hansen, Lars Peter and Scott F. Richard. 1987. The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models. *Econometrica* 50:587–614.

Hansen, Lars Peter and Thomas J. Sargent. 2001. Robust Control and Model Uncertainty. *American Economic Review* 91 (2):60–66.

Hansen, Lars Peter and Jose Scheinkman. 2009. Long-Term Risk: An Operator Approach. *Econometrica* 77:117–234.

———. 2012a. Recursive Utility in a Markov Environment with Stochastic Growth. *Proceedings of the National Academy of Sciences* 109 (30):11967–11972.

Hansen, Lars Peter and José A. Scheinkman. 2012b. Pricing Growth-Rate Risk. *Finance and Stochastics* 16:1–15.

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2008. Consumption Strikes Back?: Measuring Long-Run Risk. *Journal of Political Economy* 116:260–302.

Kreps, David M. and Evan L. Porteus. 1978. Temporal Resolution of Uncertainty and Dynamic Choice. *Econometrica* 46 (1):185–200.

Newman, C. M. and B. W. Stuck. 1979. Chernoff Bounds for Discriminating between Two Markov Processes. *Stochastics* 2 (1-4):139–153.

Ross, Stephen A. 2013. The Recovery Theorem. *The Journal of Finance* forthcoming.

Stutzer, Michael. 2003. Portfolio Choice with Endogenous Utility: A Large Deviation Approach. *Journal of Econometrics* 116:365–386.