An Appreciation of A. W. Phillips

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Introduction

A way to honor A. W. Phillips is to describe the continuing influence of one of his enduring contributions to economic dynamics, his remarkable 1959 *Biometrika* paper about how discrete time observations can be used to restrict a continuous time linear model. That paper precisely described what later came to be known as the problem of 'aggregation over time,' set forth a framework for studying it, and achieved useful characterizations of it. Phillips's 1959 paper partly shared the destiny of John F. Muth's two 1960 and 1961 papers about rational expectations. It took years for other economists to recognize how much more could be done with their ideas. In 1960, both Phillips and Muth were far ahead of most other economists in their understanding of the technicalities of time series analysis, and their appreciation for its potential applications to economic dynamics.Economists were not to take up the inquiry from the point left off by Phillips until the early 1970's, when A. R. Bergstrom, Christopher Sims, P. C. B. Phillips, and others returned to the problem of aggregation over time.

Phillips's framework

Phillips assumed that observations on a vector of data are generated by a continuous time vector stochastic process with a rational spectral density matrix.[1] Continuous time processes with rational spectral densities form a natural environment for studying the effects of aggregation over time, for several interrelated reasons that Phillips described and exploited. In particular, a rational spectral density for the continuous time process implies that:

(a) The spectral density of the discrete time stochastic process implied by skip-sampling (i.e., point-in-time sampling) the continuous time process is also rational.

(b) The autocovariance function of the continous time process is a positive semi-definite *function* defined on the real line that forms a weighted sum of exponentials, with decay parameters λ_i satisfying $|\lambda_i| < 1$.

(c) The autocovariance sequence of the discrete time process is a positive semi-definite *sequence* defined on the integers, which can be expressed as a weighted sum of geometric functions, with decay factors $\alpha_i = \exp(\lambda_i)$,

[1] For continuous time, a rational spectral density is one that can be expressed as a (matrix) *ratio* of two finite order polynomials in angular frequency ω, where for a matrix, 'division' means multiplying by its inverse.

where the λ_i's are the same as those in (b), with the *same* weights composing the continuous time autocovariance function.

(d) The continuous time data can be represented as a system of linear differential equations driven by a continuous time vector 'white noise,' itself physically 'unrealizable' as a stochastic process, but which is well defined via processes formed by convoluting it with functions in L^2.

(e) The discrete time data can be represented by a system of linear difference equations, driven by a discrete time vector white noise process.

(f) The continuous time differential equation can be solved to express the continuous time process as a convolution of a function in L^2 with the continuous time noise, i.e., as a continuous time distributed lag of the white noise.

(g) The difference equation for the discrete time data can be solved to express the data as a convolution of a sequence in ℓ^2 with the discrete time white noise.

Phillips's paper describes some of the relationships among these various types of representations, and uses them to discuss how to infer, in so far as this is possible, the parameters of the continuous time process from skip-sampled discrete time data. Phillips paper assumes that consistent estimates of the discrete autocovariance function in (c) are available. Results (b) and (c) are the foundation of Phillips's approach, which intends properly to 'fill in' the missing elements of the continuous time autocovariance function, in light of the restrictions imposed by (b) and (c). Though he doesn't complete the job of characterizing its solution, he makes a great start. The formulas that Phillips develops for (b) and (c) prepare the way for a complete analysis of the identification or 'aliasing problem' involved in inferring a continuous time model from skip-sampled discrete data. The aliasing problem is that, in general, multiple continuous time models have the same discrete-time autocovariance function. The source of multiplicity is that for any integer $k, \alpha_i = \exp(\lambda_i \pm 2\pi i k)$ satisfies the formula linking the exponential factors in (b) and (c). That there are multiple sets of λ_i's that satisfy this equation for given discrete α_i's is the beginning of the aliasing problem. P.C.B. Phillips (1972, 1973, 1974) and Hansen and Sargent (1983, 1991) pursued this aspect of Phillips's work, and in a sense completed it.

Christopher A. Sims (1971) and John Geweke (1976) took up another theme of Phillips's paper, the issue of whether and how closely discrete time distributed lags resemble continuous time distributed lags. Sims developed a powerful formula expressing the discrete time distributed lag in terms of as a convolution of the underlying continuous time distributed lag with a particular function of the continuous time autocovariance function of the regressors. Sims's formula can be interpreted as a sophisticated adaptation of Theil's omitted variable theorem, in which the 'particular function' just mentioned embodies the projection of the missing variables (in this case, a continuum of variables at the missing points between the sampling points) on the included variables (the point-in-time-sampled regressors).

Another matter taken up in recent work concerns inferring features of continuous time vector autoregressions, including innovation accountings, from their discrete time counterparts. This work is about comparing pairs of Phillips's representations, either (d) and (e), or (f) and (g). Hansen and Sargent (1991) studied situations in which discrete-time impulse response functions resemble the underlying continuous time impulse responses. Hansen and Sargent remained within Phillips's framework of a rational continuous time spectral density, and study the role of mean square continuity and differentiability of the vector stochastic process in yielding or breaking resemblence. The order of mean square differentiability for the vector process is inherited directly from the orders of the polynomials determining the rational spectral density matrix. Albert Marcet (1991) studied these questions in a framework that breaks with Phillips by dealing with continuous time processes without rational spectral densities. The assumption of a rational spectral density imposes continuity and higher-order differentiability on the kernel (i.e., the continuous time distributed lag) in representation (f). Marcet kept the kernel in L^2, but let it be discontinuous. He showed how those discontinuities substantially complicate the problem of aggregation over time, and make it possible for the discrete time impulse response functions badly to misrepresent the continuous time representations.[2]

The problem of aggregation over time was at first taken up mainly by macroeconomists interested in the practical implementation of another line of work to which A. W. Phillips contributed: the quantitative analysis and design of macroeconomic and monetary policy. Thus, because they wanted to inform monetary policy (which is executed hourly), economists at the Federal Reserve wanted to estimate weekly, daily, or even continuous time models, and approached the aliasing problem via the analysis of information processing for monetary control. Kareken, Muench, and Wallace's 1973 work on timely use of information for controlling a Keynesian macroeconomic model attributed an intricate type of 'rational expectations' to the monetary authority, but none to the private agents in the economy. That same year, Wallace and others began building macroeconomic models with private agents whose possession of rational expectations substantially, sometimes adversely, affected what the monetary authorities could achieve by way of stabilization policy.

Rational expectations threw new light over wide areas of macroeconomics and econometrics, including the aggregation over time or aliasing problem. The assumption of rational expectations often provides enough identifying information to infer a unique continuous time model from discrete time data. Christiano (1984), Christiano and Eichenbaum (1986) Heaton (1993), and Hansen and Sargent (1991) have exploited this route to identification, both in

[2] Sims (1984) deduced a 'local martingale' characterization for a class of continous time processes, a result that he uses to interpret the good fits of random walk models for a variety of asset prices. Sims's continous time processes aren't martingales, but are 'locally unpredicatable.'

the context of models with rational spectral densities. Interestingly, the rational expectations cross-equation restrictions can achieve identication without using the restriction to a rational spectral density matrix.[3]

A recent paper on aggregation over time for nonlinear models is by Hansen and Scheinkman (1995). They use moment conditions for 'test functions' to identify nonlinear continuous time Markov processes from discrete-time data (see also Ait-Sahalia 1995). Their main examples are models specified as nonlinear stochastic differential equations generalizing (d) but within the confines of Markov theory. Operator counterparts to the issues raised by Phillips under (b) and (c) recur in Hansen and Scheinkman (1995).

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[3] Christiano contributed a vivid heuristic image of how the rational expectations cross-equation restrictions can resolve the aliasing problem. A manifestation of the aliasing problem familiar to movie-goers is the illusion that wheels appear to be moving backwards when they should be moving forward, and that occasionally when a vehicle speeds up, its wheels appear first to accerlerate, then to move backwards. This illusion stems from the fact that the movie camera is skip-sampling a continuous process, leading the eye to solve the equation $\alpha_i = \exp(\lambda_i + 2\pi ik)$ for a *negative* value of α_i when the actual α_i is positive. If one focuses on the wheel alone, there is no way to determine from the skip-sampled camera shots whether or not the backward motion is an illusion. The motion of the vehicle relative to other objects (e.g., trees in the background) allows the viewer to determine the speed and direction of the vehicle, and so to resolve the aliasing problem. The motion relative to other objects plays a role similar to one exploited by the econometrician in rational expectations models, who interprets discrete-time observations on people's behavior in terms of their forecasts of the underlying continuous time stochastic process.

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