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Shock elasticities and impulse responses

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Abstract We construct shock elasticities that are pricing counterparts to impulse response functions. Recall that impulse response functions measure the importance of next-period shocks for future values of a time series. Shock elasticities measure the contributions to the price and to the expected future cash flow from changes in the exposure to a shock in the next period. They are elasticities because their measurements compute proportionate changes. We show a particularly close link between these objects in environments with Brownian information structures.

Keywords Shock elasticities · Nonlinear impulse response functions · Risk pricing · Markov dynamics · Malliavin derivative

> There are several alternative ways in which one may approach the *impulse problem* .... One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were *exposed to a stream of erratic shocks* that constantly upsets the continuous evolution, and by so doing introduces into the system the energy necessary to maintain the swings Ragnar Frish [11].

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1 Introduction

Impulse response functions characterize the impact of "a stream of erratic shocks" on dynamic economic models. They measure the consequences of alternative shocks on the future variables within the dynamical system. These methods are routinely used in linear time series analysis, and they can be generalized to nonlinear environments. See [12,19], and [13] for nonlinear extensions.

Models of asset valuation assign prices to the "stream of erratic shocks" that Frisch references. Macroeconomic shocks by their nature are not diversifiable, and as a consequence, exposure to them requires compensation. The familiar impulse response methods have counterparts in the study of valuation of stochastic cash flows within dynamic economic models. [5,16] and [14] study dynamic asset pricing through altering the cash-flow exposure to shocks. Changing this exposure alters the riskiness of the cash flow and an economic model of the stochastic discount factor gives the implied compensation. Formally, these methods construct shock-exposure and shock-price elasticities to characterize valuation as it depends on investment horizons. The elasticities are responses obtained by conveniently normalizing the exposure changes and studying the impact on the logarithms of the expected returns. These are the ingredients to risk premia, and they have a "term structure" induced by the changes in the investment horizons.

As we will show, there is a close mathematical and conceptual link between what we call shock elasticities and impulse response functions commonly used to characterize the behavior of dynamical systems. In effect the shock-price elasticities are the pricing counterparts to appropriately scaled impulse response functions. We connect these two concepts by interpreting impulse response functions and shock elasticities as changes of measure for the next-period shock.

In addition to delineating this connection, we show how continuous-time formulations with Brownian motion information structures provide additional simplicity obtained by exploiting local normality building on the Haussmann–Clark–Ocone representation of a stochastic integral of responses to past shock depicted as Brownian increments.

2 Basic setup

Let X be a Markov diffusion in $\mathbb{R}^n$:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

with initial condition $X_0 = x$. Here, $\mu(x)$ is an n-dimensional vector and $\sigma(x)$ is an $n \times k$ matrix for each vector x in $\mathbb{R}^n$. In additon W is a k-dimensional Brownian motion. We use this underlying Markov process to construct an *additive functional* Y via:

$$Y_t = Y_0 + \int_0^t \beta(X_u)du + \int_0^t \alpha(X_u) \cdot dW_u \tag{2}$$

where $\beta(x)$ is a scalar and $\alpha(x)$ is a k-dimensional vector.[1] Thus Y_t depends on the initial conditions $(X_0, Y_0) = (x, y)$ and the innovations to the Brownian motion W between dates zero and t. Let $\{\mathcal{F}_t : t \geq 0\}$ be the (completed) filtration generated by the Brownian motion. In what follows we will not explore the consequences of the initial condition $Y_0 = y$, and

[1] Notice that our definition of additive functional allows for processes of unbounded variation.

we will drop reference to y in our notation. The shock elasticities that we will formulate do not depend on this initialization.

When building models of economic time series, researchers typically work in logarithms. We think of Y as such a model, which by design can capture arithmetic growth that is stochastic in nature. Our interest in asset pricing will lead us to study exponentials of additive functionals. We call the process $M \doteq \exp(Y)$ a multiplicative functional and use it to model levels of cash flows and stochastic discount factors.

To construct an impulse response function, consider for the moment a discrete-time counterpart indexed by the length of the time period τ

$$\begin{aligned} X^{\tau}_{t+\tau} - X^{\tau}_{t} &= \mu\left(X^{\tau}_{t}\right)\tau + \sigma\left(X^{\tau}_{t}\right)\Delta W_{t+\tau} \\ Y^{\tau}_{t+\tau} - Y^{\tau}_{t} &= \beta\left(X^{\tau}_{t}\right)\tau + \alpha\left(X^{\tau}_{t}\right)\Delta W_{t+\tau}. \end{aligned} \tag{3}$$

where $\Delta W_{t+\tau} \doteq W_{t+\tau} - W_t$ and $t \in \{0, \tau, 2\tau, \ldots\}$. For convenience we may think of $\tau = 2^{-j}$ as a sequence of embedded refinements for a continuous-time approximation realized when j becomes arbitrarily large. Whenever we use the time index t in a discrete-time model with period length τ, we have in mind $t \in \{0, \tau, 2\tau, \ldots\}$.

3 Impulse response functions in discrete time

An impulse response quantifies the impact of a shock, $\Delta W_\tau = w$, on future values of Y_t. One way to construct the impulse response function is to compute

$$\Phi(t, x, w) = E\left[\phi(Y_t)|X_0 = x, \Delta W_\tau = w\right] \tag{4}$$

for alternative functions ϕ of Y_t and explore the consequences of changing a basic distribution $Q(w)$ to a "perturbed" distribution $Q^\eta(w)$. That is, we evaluate:

$$\int \Phi(t, x, w) Q^\eta(dw|x) - \int \Phi(t, x, w) Q(dw). \tag{5}$$

The multiplicative functional M is one example of a function $\phi(\cdot)$, and we will denote the conditional expectation (4) for such a multiplicative functional as Φ_m.

While the baseline distribution, Q, for the initial shock is normal with mean zero and covariance matrix τI, we may, for example, construct the perturbed distribution Q^η to explore implications of mean shifts. In this case Q^η is a normal distribution with mean $\eta(x)$, which is very similar to the suggestion of [12]. Changing η reveals the sensitivity of the predicted response to changes in the different components of ΔW_τ.[2]

Alternatively we may condition on $\Delta W_\tau = \eta(x)$ and study the response

$$\Phi(t, x, \eta(x)) - \int \Phi(t, x, w) Q(dw). \tag{6}$$

This follows an approach proposed by [19] and corresponds to measuring the response of Φ to the new information contained in the realization $\eta(x)$ of the shock ΔW_τ and exploring

[2] Gallant et al. [12] consider an impulse to the state variable, say X_0, but we can construct an analog using an impulse to the initial period shock. For a shock η, they construct

$$\Phi(t, x, w + \eta) - \Phi(t, x, w),$$

and in much of their analysis, they form averages as in (5) except that they also integrate over the initial state x.

changes in $\eta(x)$. Mathematically, this calculation is equivalent to letting Q^η assign probability one to a single value $w = \eta(x)$. That is, $Q^\eta(dw|x) = \delta[w - \eta(x)]$ where $\delta(\cdot)$ is the Dirac delta function.

The impulse response defined in (5) generally does not scale linearly with the magnitude of $\eta(x)$, so the magnitude of the impulse matters. This leads us to construct a *marginal* response. Consider a family of $Q^\eta_{\mathsf{r}}(dw|x)$ of distributions indexed by the scalar parameter r, where Q^η_{r} is normal with mean $\mathsf{r}\eta(x)$. We construct a *marginal* response by differentiating with respect to r:

$$\frac{d}{d\mathsf{r}} \int \Phi(t, x, w) Q^\eta_{\mathsf{r}}(dw|x) \bigg|_{\mathsf{r}=0} = \eta(x) \cdot \int \Phi(t, x, w) w Q(dw). \tag{7}$$

which is linear in the direction $\eta(x)$.

With any of these approaches, by freely altering ϕ, we trace out distributional responses of Y_t to a change in the distribution of the shock ΔW_τ, which is consistent with suggestions in [13].[3] Our interest lies in the continuous-time formulation. We show that in this case, the three constructed responses (5)–(7) coincide.

4 Shock elasticities in discrete time

In understanding how economic models assign values to exposures to uncertainty, we construct elasticities to changes in shock exposures. While modeling stochastic growth in terms of logarithms of economic time series is common and convenient, our interest is in asset valuation, and this leads us to an alternative but related formulation. Let G be a stochastic growth process typically representing a cash flow to be priced and S a stochastic discount factor process. We construct $\log G$ and $\log S$ in the same manner as our generic construction of the additive functional Y described previously in Eq. (2). The processes G and S are referred to as multiplicative as they are exponentials of additive functionals. They model stochastic compounding in growth and discounting in ways that are mathematically convenient. They have a common mathematical structure as does their product.

We consider both shock-exposure elasticities and shock-price elasticities. These measure the consequences of changing the exposure to a shock on hypothetical asset payoffs and prices over alternative investment horizons. As we will see, these shock elasticities resemble closely impulse response functions, but they are different in substantively important ways.

4.1 Changing the exposure

First, we explore the impact of changing the risk exposure by letting $Y = \log G$ and introducing the random variable

$$H_\tau = \exp\left[\eta(X_0) \cdot \Delta W_\tau - \frac{1}{2}|\eta(X_0)|^2 \tau\right]. \tag{8}$$

This random variable has conditional mean one conditioned on X_0. Note that

$$G_t H_\tau = \exp\left[\log G_t + \eta(X_0) \cdot \Delta W_\tau - \frac{1}{2}|\eta(X_0)|^2 \tau\right].$$

[3] Gourieroux and Jasiak [13] propose a formula similar to (5) and suggest other impulses than mean shifts, including changes in the volatility of the initial-period shock. See their formula in the middle of page 11, and the second last paragraph of their Sect. 3.2.

We thus use H_τ to increase the exposure of the stochastic growth process G to the next-period shock ΔW_τ in the direction $\eta(X_0)$. The direction vector is normalized to satisfy $E\left[|\eta(X_0)|^2\right] = 1$.

Our interest lies in comparing expected cash flows for cash flow processes with different exposures to risk. Motivated by the construction of elasticities, we focus on the proportional impact of changing the exposure expressed in terms of the ratio

$$\frac{E\left[G_t H_\tau \mid X_0 = x\right]}{E\left[G_t \mid X_0 = x\right]}$$

or expressed as the difference in logarithms

$$\log E\left[G_t H_\tau \mid X_0 = x\right] - \log E\left[G_t \mid X_0 = x\right].$$

For a marginal counterpart of this expression, we localize the change in exposure by considering a family of random variables parameterized by a ‘perturbation’ parameter r:

$$H_\tau(\mathsf{r}) = \exp\left[\mathsf{r}\eta(X_0)\cdot\Delta W_\tau - \frac{1}{2}\mathsf{r}^2|\eta(X_0)|^2\tau\right]. \tag{9}$$

Following [4] we construct the derivative

$$\left.\frac{d}{d\mathsf{r}}\log E\left[G_t H_\tau(\mathsf{r}) \mid X_0 = x\right]\right|_{\mathsf{r}=0} = \eta(x)\cdot\frac{E\left[G_t \Delta W_\tau \mid X_0 = x\right]}{E\left[G_t \mid X_0 = x\right]}$$

that represents the proportional change in the expected cash flow to a marginal increase in the exposure to the shock ΔW_τ in the direction $\eta(x)$. This leads us to label this derivative a *shock-exposure elasticity* for the cash flow process G. This elasticity depends both on the maturity of the cash flow t as well as on the current state $X_0 = x$.

4.1.1 A change of measure interpretation

The construction of the shock-exposure elasticity has a natural interpretation as a change of measure that provides a close link to the impulse response functions that we delineated in Sect. 3. Multiplication of the stochastic growth process G by the positive random variable H_τ constructed in (8) prior to taking expectations is equivalent to changing the distribution of ΔW_τ from a normal Q with mean zero and covariance τI to a normal Q^η with mean $\eta(x)$ and covariance τI. As a consequence,

$$\begin{aligned} E\left[G_t H_\tau | X_0 = x\right] &= \int \Phi_{gh}(t,x,w)Q(dw) \\ &= E\left[E\left[G_t \mid \Delta W_\tau; X_0 = x\right] H_\tau | X_0 = x\right] = \int \Phi_g(t,x,w)Q^\eta(dw|x) \end{aligned} \tag{10}$$

where the function Φ_g is defined as in (4) with $\phi(Y_t) = \exp(Y_t) = G_t$. The first row in expression (10) uses the baseline Q distribution for ΔW_τ. On the other hand, in the second row of (10) we use the perturbed distribution Q^η distribution as a computational device to alter the risk exposure of the process G.

Expression (10) relates the shock-exposure elasticity to the nonlinear impulse response functions. We compute shock elasticities by altering the exposure of the stochastic growth process G to the shock ΔW_τ, or, equivalently, by changing the distribution of this shock.

Since we are interested in computing the effects of a marginal change in exposure, we use the family H_τ (r) from Eq. (9) to define a family of measures Q^η_{r} as in (7). The shock-exposure elasticity can then be computed as

$$\frac{d}{d\mathsf{r}} \log \int \Phi_g(t,x,w) Q^\eta_{\mathsf{r}}(dw|x) \Big| = \eta(x) \cdot \frac{\int \Phi_g(t,x,w) w Q(dw)}{\int \Phi_g(t,x,w) Q(dw)}. \tag{11}$$

4.1.2 A special case

Linear vector-autoregressions (VARs) are models (3) with parameters that satisfy

$$\mu(x) = \bar{\mu} x \qquad \sigma(x) = \bar{\sigma}$$
$$\beta(x) = \bar{\beta} \cdot x \qquad \alpha(x) = \bar{\alpha}$$

X is a linear vector-autoregression with autoregression coefficient $\bar{\mu}\tau + I$ and shock exposure matrix $\bar{\sigma}$. Let $\eta(x) = \bar{\eta}$ where $\bar{\eta}$ is a vector with norm one. The impulse response function of Y_t where $t = j\tau$ for the linear combination of shocks chosen by the vector $\bar{\eta}$ is given by

$$E[Y_t \mid X_0 = x, \Delta W_\tau = \bar{\eta}] - E[Y_t \mid X_0 = x] = \bar{\kappa}_j \cdot \bar{\eta}. \tag{12}$$

where

$$\bar{\kappa}_{j+1} = \bar{\kappa}_j + \tau \bar{\beta} \bar{\zeta}_j$$
$$\bar{\zeta}_{j+1} = (I + \tau \bar{\mu}) \bar{\zeta}_j$$

with initial conditions $\bar{\zeta}_1 = \bar{\sigma}$ and $\bar{\kappa}_1 = \bar{\alpha}$. Thus

$$\bar{\zeta}_j = (I + \tau \bar{\mu})^{j-1} \bar{\sigma}$$
$$\bar{\kappa}_j = \bar{\alpha} + \left(\bar{\mu}^{-1} \left[(I + \tau \bar{\mu})^{j-1} - I \right] \bar{\sigma} \right)' \bar{\beta}.$$

The first term, $\bar{\alpha} \cdot \bar{\eta}$, represents the "instantaneous" impact arising from the current shock, while the second term captures the subsequent propagation of the shock through the dynamics of the model.

Now consider the shock elasticity for the multiplicative functional $G = \exp(Y)$. Using the formula for the expectation of a log-normal random variable, we have

$$\frac{\Phi_g(t,x,w)}{\int \Phi_g(t,x,v) Q(dv)} = \exp\left(-\frac{1}{2} |\bar{\kappa}_j|^2 + \bar{\kappa}_j \cdot w \right).$$

Using formula (11), we obtain the shock-exposure elasticity for G:

$$\bar{\eta} \cdot \int \frac{w}{\sqrt{2\pi n}} \exp\left(-\frac{1}{2} |\bar{\kappa}_j|^2 + \bar{\kappa}_j \cdot w \right) \exp\left(-\frac{1}{2} w' w \right) = \bar{\eta} \cdot \bar{\kappa}_j$$

Thus for a linear model, our shock-exposure elasticity for G coincides with the linear impulse response function for $Y = \log G$, with the direction vector $\bar{\eta}$ selecting a particular combination of shocks in ΔW_τ.

4.2 Pricing the exposure

Given our interest in pricing we are led to the study of the sensitivity of expected returns to shocks. We will utilize the family of cash flows $G_t H_\tau$ (r) indexed by the perturbation parameter r and construct a local measure of this sensitivity as the pricing counterpart of the shock-exposure elasticity (11).

A stochastic discount factor S is a stochastic process that represents the valuation of payoffs across states and time. Therefore, the value at time zero of a cash flow $G_t H_\tau(\mathsf{r})$ maturing at time t (or *cost* of purchasing an asset with such a payoff) is $E\left[S_t G_t H_\tau(\mathsf{r}) \mid X_0 = x\right]$. Since we assume that S and G are multiplicative functionals, so is their product SG.

The logarithm of the expected return for the cash flow $G_t H_\tau(\mathsf{r})$ maturing at time t is

$$\underbrace{\log E\left[G_t H_\tau(\mathsf{r}) | X_0 = x\right]}_{\text{log expected payoff}} - \underbrace{\log E\left[S_t G_t H_\tau(\mathsf{r}) | X_0 = x\right]}_{\text{log cost}}.$$

Since both components of the expected return are distorted by the same random variable $H_\tau(\mathsf{r})$, we can write the (log) expected return on cash flow $G_t H_\tau(\mathsf{r})$ as

$$\log \int \Phi_g(t, x, w) Q_{\mathsf{r}}^{\eta}(dw|x) - \log \int \Phi_{sg}(t, x, w) Q_{\mathsf{r}}^{\eta}(dw|x).$$

We localize the change in exposure by computing the derivative of this expression with respect to r and evaluating this derivative at $\mathsf{r} = 0$. This calculation yields the discrete-time *shock-price elasticity*

$$\eta(x) \cdot \frac{\int \Phi_g(t, x, w) w Q(dw)}{\int \Phi_g(t, x, w) Q(dw)} - \eta(x) \cdot \frac{\int \Phi_{sg}(t, x, w) w Q(dw)}{\int \Phi_{sg}(t, x, w) Q(dw)}. \tag{13}$$

The shock-price elasticity is the difference between a shock-exposure elasticity and a shock-cost elasticity. Locally, the impulse response for both components of the shock-price elasticity is given by the covariance of the impulse response Φ with the shock ΔW_τ. In what follows we will show how a continuous-time formulation gives us an alternative way to localize shock exposures in a convenient way.

The shock-price elasticity for the one-period horizon has particularly simple representation and is given by:

$$\eta(x) \cdot \alpha_g(x) - \eta(x) \cdot \left[\alpha_g(x) + \alpha_s(x)\right] = \eta(x) \cdot \left[-\alpha_s(x)\right].$$

In this formula the entries of $-\alpha_s(x)$ give the vector of "prices" assigned to exposures of each of the entries of ΔW_τ. These entries are often referred to as the (local) price of risk. Our shock-price elasticity function captures the term structure of the price of risk, in the same way as an impulse response function captures the dynamic adjustment of an economy over time in response to a shock.

In the special case of a log-normal model discussed in Sect. 4.1.2, we can construct a model of the stochastic discount factor $S = \exp(Y)$. In this case, the linearity of the results implies that the shock-price elasticity (13) will correspond to the impulse response function for $-\log S$.

5 Returning to continuous time

By taking continuous-time limits, we achieve some simplicity given the Brownian motion information structure. In this section, we proceed informally to provide economic intuition. A more formal treatment follows in Sect. 6.

Given a Markov diffusion X such as (1), the Malliavin derivative $\mathcal{D}_0 X_t$ allows us to examine the contribution of a shock dW_0 to the value of that diffusion at time $t > 0$. We calculate the Malliavin derivative recursively by computing what is called the *first variation* process associated to the diffusion.[4] The first variation process for X is an $n \times n$ -dimensional process Z^x that measures the impact of the change in initial condition $X_0 = x$ on future values of the process X:

$$Z_t^x = \frac{\partial X_t}{\partial x'}.$$

This process solves a linear stochastic differential equation obtained by differentiating the coefficients:

$$dZ_t^x = \left[\frac{\partial}{\partial x'} \mu(X_t) \right] Z_t^x dt + \sum_{i=1}^{k} \left(\frac{\partial}{\partial x'} [\sigma\,(X_t)]_{\cdot i} \right) Z_t^x dW_{i,t} \tag{14}$$

where $[\sigma\,(x)]_{\cdot i}$ is the i-th column of $\sigma\,(x)$ and W_i is the i-th component of the Brownian motion. The initial condition for the first variation process is $Z_0^x = \mathbf{I}_n$. The Malliavin derivative satisfies

$$\mathcal{D}_0 X_t = Z_t^x \sigma\,(x)\,,$$

since $\sigma\,(x)$ gives the impact of the shock dW_0 on dX_0.

The construction of the Malliavin derivative can be extended to the additive functional Y. Since (X, Y) form a Markov diffusion, and the coefficients in this Markov diffusion are independent of Y, we obtain a $1 \times n$ process Z^y that satisfies:

$$dZ_t^y = \left[\frac{\partial}{\partial x'} \beta(X_t) \right] Z_t^x dt + \sum_{i=1}^{k} \left(\frac{\partial}{\partial x'} \alpha_i\,(X_t) \right) Z_t^x dW_{i,t} \tag{15}$$

with initial condition $Z_0^y = \mathbf{0}_{1\times n}$. Consequently, the Malliavin derivative of Y is an $1 \times k$ vector given by

$$\mathcal{D}_0 Y_t = Z_t^y \sigma\,(x) + \alpha\,(x)'$$

where $\alpha\,(x)$ represents the initial contribution of the shock $\eta(x) \cdot dW_0$ to Y_t and $Z_t^y \sigma\,(x)\,\eta(x)$ captures the propagation of the shock through the dynamics of the state vector X.

When conditional mean coefficients (μ, β) are linear in state vector, the exposure coefficients (σ, α) on the Brownian increment are constant, and the η vector is constant, then the Malliavin derivatives only depend on the date zero state and not on the Brownian increments that follow. In this case the Malliavin derivative calculations will yield the continuous time counterpart to the calculations in Sect. 4.1.2. More generally, the Malliavin derivatives depend on the Brownian increments. We could compute the "average" responses using

$$\Phi_y(t, x) = \eta(x) \cdot E\,[\mathcal{D}_0 Y_t \mid X_0 = x]$$

for $t \geq 0$ which still depends on the initial state x but not on the Brownian increments.

We have featured η as a device for selecting which (conditional) linear combination of increments to target for the response function Φ_y. In fact, Malliavin derivatives are typically computed by introducing drift distortions for the Brownian increment vector dW_0. Thus an equivalent interpretation of the role of η in computation of Φ_y is that of a date zero drift distortion, the counterpart to the mean shift for a normally distributed random vector that we used in our discrete-time constructions.

[4] See e.g. Property P2 on page 395 of [10]. Fournié et al. [10] assume that the diffusion coefficients have bounded derivatives, which is not verified in this example. A precise justification would require extending their theorem to our setup.

Since we are interested in the process $M = \exp(Y)$, averaging the random responses of $\log M$ will not be of direct interest in our analysis of elasticities. This leads us to modify our calculation of average responses.

The Malliavin derivative of the function of a process $U_t = \phi(Y_t)$ is well defined provided the function ϕ is sufficiently regular. In this case we may use a "chain-rule,"

$$\mathcal{D}_u U_t = \phi'(Y_t)\mathcal{D}_u Y_t.$$

The function $\phi = \exp$ is of particular interest to us since $M = \exp(Y)$. For the shock $\eta(x) \cdot dW_0$,

$$\mathcal{D}_0 M_t \eta(x) = \exp(Y_t)\mathcal{D}_0 Y_t \eta(x)$$

gives the date t distributional response of M_t to the date zero shock $\eta(x) \cdot dW_0$.

Next we construct a nonlinear counterpart to a moving-average representation, which for a continuous-time diffusion is the Haussmann–Clark–Ocone representation. The Haussmann–Clark–Ocone representation uses Malliavin derivatives to produce a "moving-average" representation of M with state-dependent coefficients and is typically expressed as:

$$M_t = \int_0^t E\left[\mathcal{D}_u M_t \mid \mathcal{F}_u\right] \cdot dW_u + E[M_t]. \tag{16}$$

Many of the random variables we consider depend on $X_0 = x$ along with the Brownian motion W, including random variables on both sides of (16). We hold $X_0 = x$ fixed for the calculation of the Malliavin derivatives, and rewrite Eq. (16) as:

$$M_t = \int_0^t E\left[\mathcal{D}_u M_t \mid \mathcal{F}_u, X_0 = x\right] \cdot dW_u + E[M_t \mid X_0 = x]. \tag{17}$$

The notation $E[\,\cdot \mid X_0 = x]$ should remind readers that the computation of the expectation over the function of the Brownian motion depends on the choice of initial conditions.

This convenient result represents M_t as a response to shocks with "random coefficients" $E\left[\mathcal{D}_u M_t \mid \mathcal{F}_u, X_0 = x\right]$ that are adapted to $\mathcal{F}_u$ whereas in linear time series analysis these coefficients are constant. With representation (17), a continuous-time analogue to the impulse response functions computed in Sect. 3 measures the impact on $\phi(Y_t)$ of a "shock" dW_0:

$$\Phi_m(t, x) = \eta(x) \cdot E\left[\exp(Y_t)\mathcal{D}_0 Y_t \mid X_0 = x\right], \tag{18}$$

for $t \geq 0$. The term $\mathcal{D}_0 Y_t$ can be computed using the recursive calculations outlined above. The weighting by the nonstationary process $M = \exp(Y)$ may be important, because M grows or decays stochastically over time.

Next we consider shock elasticities in continuous time. In this paper we build these elasticities in a way that is consistent with those given in [5], but we derive them in a more direct way.[5] In Sect. 6 we will show that the elasticities of interest can be expressed as

[5] Borovička et al. [5] consider responses over finite investment intervals and introduce a separate parameter that localizes the risk exposure. Similarly, [4] use a discrete time economic environment and again introduce a parameter that localizes the risk. Here we avoid introducing an additional parameter by letting the continuous-time approximation localize the risk exposure over arbitrarily short time intervals. In Sect. 8 we elaborate on the connection between calculation in this paper and our previous work [5].

$$\varepsilon(t,x) = \eta(x) \cdot \frac{E\left[\exp(Y_t)\mathcal{D}_0 Y_t \mid X_0 = x\right]}{E\left[\exp(Y_t) \mid X_0 = x\right)} \tag{19}$$

where $Y = \log G$ in the case of a *shock-exposure* elasticity and $Y = \log S + \log G$ in the case of a *shock-cost* elasticity. The numerator is the same as the impulse response for $\exp(Y_t)$ given in (18). Consistent with our interest in elasticities, we divide by the conditional expectation of $\exp(Y_t)$. In accordance with this representation, the elasticities we justify are weighted averages of the impulse responses for Y weighted by $\exp(Y)$. Asymptotic results for $t \to \infty$ can be obtained using a martingale decomposition of the multiplicative functional M analyzed in [15].

A *shock-price* elasticity is given by the difference between an exposure elasticity and a cost elasticity:

$$\eta(x) \cdot \frac{E\left[G_t\left(\mathcal{D}_0 \log G_t\right) \mid X_0 = x\right]}{E\left[G_t \mid X_0 = x\right]} - \eta(x) \cdot \frac{E\left[S_t G_t\left(\mathcal{D}_0 \log S_t + \mathcal{D}_0 \log G_t\right) \mid X_0 = x\right]}{E\left[S_t G_t \mid X_0 = x\right]}. \tag{20}$$

In a globally log-normal model $\mathcal{D}_0 \log G_t$ and $\mathcal{D}_0 \log S_t$ depend on t but are not random, and the weighting by either G_t or by $S_t G_t$ is of no consequence. Moreover, in this case the shock-price elasticities can be computed directly from the impulse response function for $-\log S$ to the underlying shocks since the expression in (20) is equal to $-\mathcal{D}_0 \log S_t$. The resulting elasticities are the continuous time limits of the results from the log-normal example introduced in Sect. 4.1.2.

Instead of computing directly the Malliavin derivatives, there is a second approach that sometimes gives a tractable alternative to computing the coefficients of the Haussmann–Clark–Ocone representation for M. This approach starts by computing $E\left[M_t \mid X_0 = x\right]$ and then differentiating with respect to the state:

$$\sigma(x)' \frac{\partial}{\partial x} E\left[M_t \mid X_0 = x\right].$$

The premultiplication by $\sigma(x)'$ acts as a measure of the local response of X to a shock. As in [5] expression (19) can be written as[6]

$$\varepsilon(t,x) = \eta(x) \cdot \left[\alpha(x) + \sigma(x)' \frac{\partial}{\partial x} \log E\left[M_t \mid X_0 = x\right]\right]. \tag{21}$$

This result separates the ‘instantaneous’ effect of the change in exposure, $\alpha(x)$, from the impact that propagates through the nonlinear dynamics of the model, expressed by the second term in the bracket. In case of the shock-price elasticity, $\alpha(x)$ corresponds to the local price of risk.

6 Formal construction of shock elasticities

To construct the elasticities that interest us, we “perturb” the cash flows in alternative ways. Let N^τ be

$$N_t^\tau = \int_0^{\tau \wedge t} \eta(X_u) \cdot dW_u.$$

6 For example see the formulas (5) and (6) and the discussion in Sect. 4 provided in [5].

The process N^τ alters the exposure over the interval $[0, \tau]$ and remains constant for $t \geq \tau$. We impose

$$E\left[|\eta(X_0)|^2\right] = 1$$

when X is stationary and X_0 is initialized at the stationary distribution. While the finite second moment condition is a restriction, our choice of unity is made as a convenient normalization. We impose this second moment restriction to insure that N^τ has a finite variance τ for $t \geq \tau$. We use the vector η to select alternative risk exposures. Let $\mathcal{E}(N^\tau)$ denote the stochastic exponential of N^τ:

$$\begin{aligned} H_t^\tau &= \mathcal{E}_t(N^\tau) = \exp\left(N_t^\tau - \frac{1}{2}[N^\tau, N^\tau]_t\right) \\ &= \exp\left[\int_0^{\tau\wedge t} \eta(X_u)\cdot dW_u - \frac{1}{2}\int_0^{\tau\wedge t} |\eta(X_u)|^2 du\right]. \end{aligned}$$

Our assumption that X is stationary with a finite second moment guarantees that the process H^τ as constructed is a local martingale. We will in fact assume that the stochastic exponentials of the perturbations N^τ are martingales. This normalization will be of no consequence for our shock-price elasticity calculations, but it is a natural scaling in any event.

Form the perturbed payoff GH^τ. This payoff changes the exposure of the payoff G by altering the shock exposure of $\log G_t$ to be

$$\int_0^t \alpha_g(X_u)\cdot dW_u + \int_0^\tau \eta(X_u)\cdot dW_u$$

for $\tau < t$. This exposure change is small for small τ. Construct the logarithm of the expected return:

$$\varepsilon_p(\tau, t, x) = \underbrace{\log E\left[G_t H_t^\tau | X_0 = x\right]}_{\text{log expected payoff}} - \underbrace{\log E\left[S_t G_t H_t^\tau | X_0 = x\right]}_{\text{log cost}}.$$

Note that

$$\varepsilon_p(\tau, t, x) = \varepsilon_e(\tau, t, x) - \varepsilon_c(\tau, t, x)$$

where

$$\begin{aligned} \varepsilon_e(\tau, t, x) &= \log E\left[G_t H_t^\tau | X_0 = x\right] \\ \varepsilon_c(\tau, t, x) &= \log E\left[G_t S_t H_t^\tau | X_0 = x\right]. \end{aligned}$$

We will calculate derivatives of ε_e and ε_c to compute a shock-price elasticity. The first of these derivatives is a shock-exposure elasticity.

We use H^τ to change the exposure to uncertainty. Since H^τ is a positive martingale with a unit expectation, equivalently it can be used as a change in probability measure. Under this interpretation, think of $\eta(X_t)$ as being a drift distortion of the Brownian motion W. Then

$$dW_t = \begin{cases} \eta(X_t)dt + d\widetilde{W}_t & 0 \leq t \leq \tau \\ d\widetilde{W}_t & t > \tau \end{cases}$$

where $\widetilde{W}$ is a Brownian motion under the change of measure. This gives an alternative interpretation to our calculations and a formal link to Malliavin differentiation. Bismut [3] uses the change of measure to perform calculations typically associated with Malliavin differentiation.

In what follows we will shrink the interval $[0, \tau]$ to focus on the instantaneous change in exposure, which we can think of equivalently as an instantaneous drift distortion in the Brownian motion.

6.1 Haussmann–Clark–Ocone formula

To characterize the derivatives of interest, we apply the Haussmann–Clark–Ocone formula (17) to $M = G$ or $M = SG$ and represent M_t as a stochastic integral against the underlying Brownian motion. The vector of state dependent impulse response functions for M_t for the date zero Brownian increment is $E\left[\mathcal{D}_0 M_t | X_0 = x\right]$ when viewed as a function of t. This averages over the random impacts in the future but still depends on $X_0 = x$.

We will also use the following result that is a consequence of Proposition 5.6 in [22]:

$$\mathcal{D}_u E\left(M_t | \mathcal{F}_\tau, X_0 = x\right) = E\left(\mathcal{D}_u M_t | \mathcal{F}_\tau, X_0 = x\right) \mathbf{1}_{[0,\tau]}(u). \tag{22}$$

6.2 Computing shock elasticities

Fix $X_0 = x$ and $\tau \leq t$. The Haussmann–Clark–Ocone formula (17) implies that:

$$\begin{aligned} E\left[E(M_t|\mathcal{F}_\tau, X_0 = x) H_t^\tau | X_0 = x\right] = E\Bigg[& H_\tau^\tau \int_0^t E\left(\mathcal{D}_u E(M_t|\mathcal{F}_\tau)|\mathcal{F}_u, X_0 = x\right) \\ & \times dW_u | X_0 = x] \\ & + E\left[H_\tau^\tau E(E(M_t|\mathcal{F}_\tau)|X_0 = x)|X_0 = x\right]. \end{aligned}$$

Hence, using Eq. (22),

$$E\left(M_t H_t^\tau | X_0 = x\right) = E\left[H_\tau^\tau \int_0^\tau E\left(\mathcal{D}_u M_t | \mathcal{F}_u, X_0 = x\right) \cdot dW_u | X_0 = x\right] + E\left(M_t | X_0 = x\right)$$

for $t \geq \tau$. Under the change of measure implied by H^τ,

$$E\left(M_t H_t^\tau | X_0 = x\right) = \widetilde{E}\left[\int_0^\tau Z_u \cdot \eta(X_u) du | X_0 = x\right] + E\left(M_t | X_0 = x\right)$$

where $\widetilde{E}$ is the expectation under the change in probability measure and

$$Z_u = E\left(\mathcal{D}_u M_t | \mathcal{F}_u, X_0 = x\right).$$

We compute the derivative with respect to τ at $\tau = 0$ by evaluating:

$$\begin{aligned} & \lim_{\tau \downarrow 0} \frac{1}{\tau} E\left[H_\tau^\tau \int_0^\tau E\left(\mathcal{D}_u M_t | \mathcal{F}_u, X_0 = x\right) \cdot dW_u | X_0 = x\right] \\ & = \lim_{\tau \downarrow 0} \frac{1}{\tau} \widetilde{E}\left[\int_0^\tau Z_u \cdot \eta(X_u) du | X_0 = x\right] \\ & = Z_0 \cdot \eta(x) \\ & = E\left(\mathcal{D}_0 M_t | X_0 = x\right) \cdot \eta(x) \\ & = E\left[M_t\left(\mathcal{D}_0 \log M_t\right) | X_0 = x\right] \cdot \eta(x) \end{aligned}$$

where the last line follows from the formula:

$$\mathcal{D}_0 M_t = M_t \mathcal{D}_0 \log M_t.$$

For globally log-normal models $\mathcal{D}_0 \log M_t$ depends only on t and not on random outcomes, but more generally $\mathcal{D}_0 \log M_t$ is random.

Since our aim is to compute elasticities, the actual differentiation that interests us is the derivative of the logarithm of the conditional expectation:

$$\varepsilon_m(t,x) = \eta(x) \cdot \left(\frac{E\left[M_t\left(\mathcal{D}_0 \log M_t\right) | X_0 = x\right]}{E\left[M_t | X_0 = x\right]} \right).$$

Thus the elasticities are weighted averages of $\mathcal{D}_0 \log M_t$ weighted by M_t.

Remark 6.1 While our focus has been on shock elasticities and impulse response functions, there is also a nice connection to a temporal dependence measure suggested by [24]. His predictive dependence measure is based on the expected consequences of changing shock distributions through a coupling. His measure is constructed in discrete time, but the L^2 version of the continuous-time analog for the stochastic formulation we use is

$$\omega(\tau) = \sqrt{2E\left[|\mathcal{D}_0 \log M_t|^2\right]}. \tag{23}$$

It is perhaps best to think of [24]'s analysis as directly applying to the X process. But his aim is to use these measures to study when central limit approximations are appropriate. In fact the additive functional $\log M$ will often obey a Central Limit Theorem and the finite limiting behavior of $\omega(\tau)$ as $\tau \to \infty$ in (23) suggests that standard martingale approximation methods are applicable.

7 Example: Persistent components in consumption dynamics

We illustrate the construction of shock elasticities using an example featured in [5]. We outline two calculations. First, we utilize the construction of the Malliavin derivative $\mathcal{D}_0 M_t$ for the multiplicative functionals of interest using the recursive calculations outlined in Sect. 5. Then we show how to compute the shock elasticities using semi-analytical formulas for the conditional expectations of the multiplicative functional.

We assume the date t state vector takes the form $X_t' = \left(X_{1,t}', X_{2,t}\right)$ where $X_{1,t}$ is an n-dimensional state vector and $X_{2,t}$ is a scalar. The dynamics of X in (1) are specified by

$$\mu(x) = \bar{\mu}(x - \iota) \qquad \sigma(x) = \sqrt{x_2}\bar{\sigma}$$

where

$$\bar{\mu} = \begin{bmatrix} \bar{\mu}_{11} & \bar{\mu}_{12} \\ 0 & \bar{\mu}_{22} \end{bmatrix} \qquad \bar{\sigma} = \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix}, \tag{24}$$

$\bar{\mu}_{11}$ and $\bar{\mu}_{12}$ are $n \times n$ and $n \times 1$ matrices, $\bar{\mu}_{22}$ a scalar, and $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are $n \times k$ and $1 \times k$ matrices, respectively. In this model, $X_{1,t}$ represents predictable components in the growth rate of the multiplicative functional, and $X_{2,t}$ captures the contribution of stochastic volatility. The vector ι is specified to be the vector of means in a stationary distribution. We set the mean of X_2 to be one in our calculations. The parameters of the additive functional Y in (2) are:

$$\beta(x) = \bar{\beta}_0 + \bar{\beta}_1 \cdot (x - \iota) \qquad \alpha(x) = \sqrt{x_2}\bar{\alpha}. \tag{25}$$

In the illustrative economic models, the parameters (α_c, β_c) governing the evolution of the logarithm of consumption $\log C$ and (α_s, β_s) governing the evolution of the logarithm of the stochastic discount factor $\log S$ have the functional form of (α, β) specified for Y above. We choose the shock selection vector $\eta(x)$ to be $\eta(x) = \sqrt{x_2}\bar{\eta}$. The vector $\bar{\eta}$ has unit norm and, for instance, can be a coordinate vector.

7.1 Construction of shock elasticities

The recursive construction of the Malliavin derivative applies formulas (14)–(15):

$$
\begin{aligned}
dZ_t^x &= \bar{\mu} Z_t^x dt + \frac{1}{2\sqrt{X_{2,t}}} (\bar{\sigma} dW_t) Z_{2,t}^x \\
dZ_t^y &= \bar{\beta}_1' Z_t^x dt + \frac{1}{2\sqrt{X_{2,t}}} Z_{2,t}^x (\bar{\alpha} \cdot dW_t).
\end{aligned}
$$

Recall that $X_{2,t}$ is the last entry of X_t. Accordingly, the row vector $Z_{2,t}^x$ is the last row of the first variation matrix Z_t^x. The initial conditions are $Z_0^x = I$ and $Z_0^y = 0$. The Malliavin derivative for each $t \geq 0$ is the $1 \times k$ vector:

$$\mathcal{D}_0 Y_t = \sqrt{x_2} \left(Z_t^y \bar{\sigma} + \bar{\alpha}' \right).$$

In the state dynamics, the X_2 process does not feedback onto the X_1 process. The stochastic differential equation for the X_2 process can thus be solved without simultaneously solving for the X_1 process. It follows from this "triangular" structure that the first n elements of the vectors $Z_{2,t}^x$ and Z_t^y are zero. Consequently, the first n elements of the first variation process Z^y are deterministic functions of time. In contrast to a log-linear environment, the last columns of processes Z^x and Z^y used to construct the Malliavin derivatives now depend on the Brownian motion because of the role of the stochastic volatility in the state dynamics. One strategy for computation is to simulate simultaneously (X, Y, Z^x, Z^y) and to approximate conditional expectations using Monte Carlo techniques.[7]

For this parameterization the other approach mentioned in Sect. 5 is tractable because we know the functional form for $E[M_t \mid X_0 = x]$ in formula (21). Results from [8] and [14] show that under appropriate parametric restrictions

$$\log E[M_t \mid X_0 = x] = \theta_0(t) + \theta_1(t) \cdot x_1 + \theta_2(t) x_2 \tag{26}$$

where the coefficients $\theta_i(t)$ satisfy a set of ordinary differential equations given in Appendix 1. Given this formula, we may directly compute the coefficients for the Haussman–Clark–Ocone representation, and thus

$$\varepsilon(x, t) = \eta(x) \cdot \left[\bar{\sigma}_1' \theta_1(t) + \bar{\sigma}_2' \theta_2(t) + \bar{\alpha} \right] \sqrt{x_2}. \tag{27}$$

7.2 Comparing two example economies

We compare two specifications of investors' preferences. The first specification (BL) endows the representative investor with time-separable, constant relative risk aversion utility as in [6] and [21]. The stochastic discount factor for the investor is given by

$$d \log S_t = -\delta dt - \gamma d \log C_t = -\left[\delta + \gamma \beta_c(X_t) \right] dt - \gamma \alpha_c(X_t) \cdot dW_t \tag{28}$$

where δ is the time-preference coefficient and γ is the risk aversion parameter. The stochastic discount factor is thus a multiplicative functional with parameters $\beta_s^{BL}(x)$ and $\alpha_s^{BL}(x)$ specified in (25).

[7] For longer investment horizons, it would likely be beneficial to change probability measures using, for instance, the martingale featured in [15]. Such an approach could better center the simulations. It could build on methods for rare event simulation, which could be valuable here because the M process grows or decays asymptotically at an exponential rate.

The second model of investors' preferences (EZ) is the recursive utility specification of the [20] and [9] type, analyzed in continuous time by [7]. These preferences allow the separation of risk aversion to intratemporal bets from intertemporal elasticity of substitution (IES). We will focus on the special case of unitary IES as this case allows us to derive semi-analytical solutions.[8] In this case,

$$d\log S_t = -\delta dt - d\log C_t + d\log \widetilde{S}_t = \beta_s^{EZ}(X_t)\,dt + \alpha_s^{EZ}(X_t)\cdot dW_t \tag{29}$$

where $\widetilde{S}$ is a multiplicative martingale satisfying

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = \sqrt{X_{2,t}}\,(1-\gamma)\left(\bar{\alpha}_c + \bar{\sigma}_1'\bar{v}_1 + \bar{\sigma}_2'\bar{v}_2\right)\cdot dW_t$$

with coefficients $\bar{v}_1, \bar{v}_2$ derived in Appendix 1. This martingale is the additional contribution coming from the continuation value for recursive utility. Equivalently, it can be interpreted as a change of measure and can be motivated as a robustness adjustment for model misspecification as suggested by [1]. Under plausible parameterizations, the Brownian motion W has a negative drift under the change of measure, reflecting the risk adjustment arising from recursive-utility or a concern for model misspecification.

A similar specification of the consumption dynamics coupled with the recursive preference structure was utilized by [2,18] and others to generate large and volatile prices of risk. We adopt the parameterization from [17] used in [5] and summarized in the caption of Figure 1. The stationary distribution for the process X_2 is a gamma distribution, which allows us to compute quartiles for shock elasticities (27) in a semi-analytical form.

Figure 1 plots the shock-exposure elasticity functions for the consumption process C and shock-price elasticity functions for the same process in the BL and EZ models.[9] We parameterize the vectors $\bar{\sigma}_1, \bar{\sigma}_2$ and $\bar{\alpha}_c$ as orthogonal, so that each component of the Brownian motion W corresponds to an orthogonal shock to the processes X_1, X_2 and C, respectively.

The zero-horizon limits of the shock elasticity functions correspond to what is known as infinitesimal exposures and infinitesimal prices of risk. The shock-exposure elasticities are equal to zero at $t = 0$ except for the consumption exposure elasticity because shocks to the growth rate and volatility processes impact the consumption process only indirectly, through changes in the levels of the processes X_1 and X_2. While a current shock to the processes X_1 or X_2 will have a persistent effect on these processes which will accumulate to a nontrivial impact on the future values of the consumption process C, the infinitesimal contribution to C is zero.

The same logic applies to the shock-price elasticities for the BL model, since the stochastic discount factor (28) for this model merely scales the consumption process by the negative of the risk aversion, $-\gamma$, and the investor thus only fears risk embedded in the consumption process that is contemporaneous with the maturity of the cash flow. On the other hand, the EZ model leads to nonzero shock-price elasticities even at the zero horizon. When an investor endowed with EZ preferences evaluates the risk embedded in the consumption process, she also fears the impact of the current shock on the consumption process beyond her investment horizon. The recursive nonseparable preference structure (29) of the EZ model, explained in

[8] Numerical calculations confirm that for this endowment economy, the shock elasticities are not very sensitive to the choice of the IES parameter.

[9] The shock-price elasticity for this parameterization of the consumption dynamics ceases to exist for long investment horizons for risk aversion coefficients $\gamma > \bar{\gamma}^{BL} \sim 20$ in the BL model due to the non-existence of the conditional expectation $E[S_t C_t \mid X_0 = x]$. Similarly, the shock-price elasticity function does not exist in the EZ model for $\gamma > \bar{\gamma}^{EZ} \sim 24$ due to the non-existence of the continuation value for the recursive preference structure. For details, see Appendix 1.

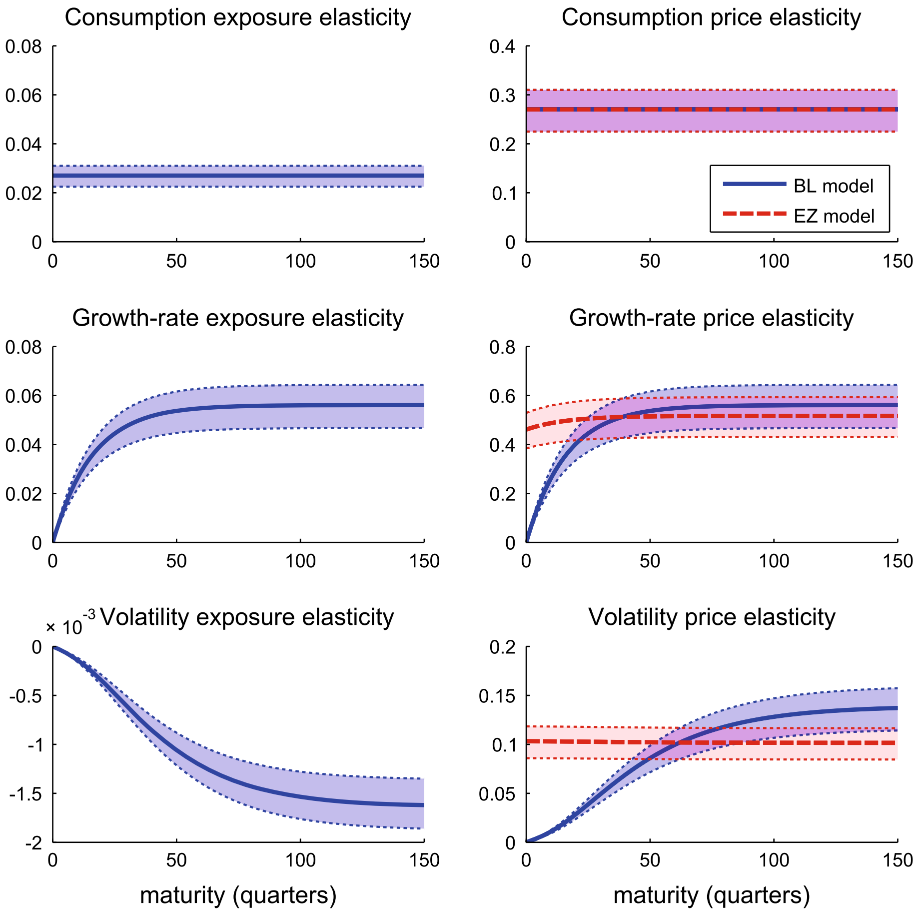

Fig. 1 Shock elasticities for the long-run risk model. *Thick lines* correspond to shock elasticities conditional on $X_{2,0} = 1$, the shaded areas capture quartiles for the shock elasticities under the stationary distribution of X_2. In the *top right panel*, the shock-price elasticities for the BL and EZ models coincide. Parameters are calibrated to monthly frequency, and the elasticities are annualized. The parameterization is $\bar{\beta}_{c,0} = 0.0015$, $\bar{\beta}_{c,1} = 1$, $\bar{\beta}_{c,2} = 0$, $\bar{\mu}_{11} = -0.021$, $\bar{\mu}_{12} = \bar{\mu}_{21} = 0$, $\bar{\mu}_{22} = -0.013$, $\bar{\alpha}_c = [0.0078\ \ 0\ \ 0]'$, $\bar{\sigma}_1 = [0\ \ 0.00034\ \ 0]$, $\bar{\sigma}_2 = [0\ \ 0\ \ -0.038]$, $\iota_1 = 0$, $\iota_2 = 1$, $\delta = 0.002$, $\gamma = 10$

detail in Appendix 1, leads to a compensation for risk over an infinitesimal horizon induced by fluctuations in future consumption realizations. The growth-rate and volatility shocks thus generate nonzero infinitesimal price elasticities through their impact on future consumption levels.

The consumption exposure elasticity measures the sensitivity of expected consumption to a direct shock to the consumption process. The elasticity function is flat, reflecting the fact that $\bar{\alpha}_c \cdot dW_t$ is an iid growth shock to the consumption process. The shaded area represents the quartiles of the stationary distribution of the elasticity function, and captures the dependence of the magnitude of the response on the current volatility level x_2.

On the other hand, the growth-rate exposure elasticity, which represents the shock to the process X_1, builds up over time as the perturbation of the persistent growth rate accumulates in the level of the consumption process. For the volatility exposure elasticity, the negative coefficient in $\bar{\sigma}_2$ implies that a positive shock reduces the volatility of the consumption process and, because of Jensen inequality, decreases the expected level of future consumption.

The right column in Fig. 1 displays the shock-price elasticity functions for the BL and EZ models. As explained before, the shock-price elasticities for the BL model approximately correspond to the shock-exposure elasticities for the consumption process scaled by the risk aversion coefficient γ. Moreover, the consumption price elasticities coincide for the BL and EZ models. Consequently, the differences in asset pricing implications of the two models must arise from exposure to the growth-rate and volatility shocks, which have predictable components. The nonseparability in the EZ preference specification is inconsequential for iid growth rate shocks.

The EZ model produces shock-price elasticity functions that are roughly flat and converge to long-term limits that are lower than those for the BL model with the same value of the risk aversion parameter. The flatness is caused by the fact that the martingale component (29) in the EZ stochastic discount factor is the dominant source of its volatility. The long-horizon shock elasticities in the two models coincide in the limit as the time preference coefficient δ declines to zero. In the EZ model, the volatility of the stochastic discount factor depends on the magnitude of the continuation values, and a decline in time discounting magnifies these continuation values so that shock-price elasticities increase.

8 Implications for changes over finite horizons

Our earlier work, [14] and [16], took a different limit. We considered "small" exposures over an entire interval. We investigate again this formulation and study the relationship between these approaches.

Consider

$$N_t^\tau(\mathsf{r}) = \mathsf{r}\int_0^{\tau\wedge t} \eta(X_u)\cdot dW_u$$

and the stochastic exponential:

$$H_t^\tau(\mathsf{r}) = \mathcal{E}_t\left(N^\tau(\mathsf{r})\right).$$

This is a generalization of the perturbation H_t^τ introduced in Sect. 6. The scalar r parameterizes the magnitude of the exposure. For $t \geq \tau$ write heuristically,

$$E\left(M_t H_t^\tau(\mathsf{r}) \mid X_0 = x\right) = E\left[M_t \int_0^\tau H_u^\tau(\mathsf{r})\left(\frac{dH_u^\tau(\mathsf{r})}{H_u^\tau(\mathsf{r})}\right) \mid X_0 = x\right].$$

Apply Law of Iterated Expectations and first compute

$$\begin{aligned}&\lim_{v\downarrow 0}\frac{1}{v}\left\{E\left[\frac{M_t H_{u+v}^\tau(\mathsf{r})}{M_u H_u^\tau(\mathsf{r})} \mid X_u\right] M_u H_u^\tau(\mathsf{r}) - E\left[M_t H_u^\tau(\mathsf{r}) \mid X_u\right]\right\}\\&\quad = \mathsf{r}\varepsilon_m(t-u, X_u) E\left[M_t \mid \mathcal{F}_u\right] H_u^\tau(\mathsf{r}).\end{aligned} \tag{30}$$

Thus

$$E\left(M_t H_t^\tau(\mathsf{r})\, X_0 = x\right) = \mathsf{r}E\left[M_t \int_0^\tau \varepsilon_m(t-u, X_u) H_u^\tau(\mathsf{r})\, du \mid X_0 = x\right]. \tag{31}$$

Instead of localizing τ, [16] differentiate the logarithm of the expression on the right-hand side with respect to r and evaluate the derivative at $\mathsf{r} = 0$. This results in:

$$\frac{E\left[M_t \int_0^\tau \varepsilon_m(t-u, X_u)du \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}$$

consistent with the formula given in [5]. The resulting shock price elasticity is:

$$\frac{E\left[G_t \int_0^\tau \varepsilon_g(t-u, X_u)du \mid X_0 = x\right]}{E\left[G_t \mid X_0 = x\right]} - \frac{E\left[S_t G_t \int_0^\tau \varepsilon_{s+g}(t-u, X_u)du \mid X_0 = x\right]}{E\left[S_t G_t \mid X_0 = x\right]}.$$

Thus we have shown that the implied interval τ elasticity is constructed from weighted averages of integrals of the continuous-time elasticities when we localize the risk over this interval by making r small.

9 Conclusion

Impulse response functions are commonly used in economic dynamics. They measure the impact of shocks on endogenously determined and exogenously specified processes in a dynamical system. We study continuous-time, nonlinear counterparts by building on the state-dependent moving average representations implied by the Haussmann–Clark–Ocone formula.

Structural models of macroeconomics typically include a stochastic discount factor process used to represent asset valuation. In this paper we studied pricing counterparts to impulse response functions. We call these counterparts shock elasticities. Exposure elasticities measure how responsive future expected cash flows are to shocks and price elasticities measure how responsive expected returns are to shocks. The shock elasticities reveal implications of stochastic equilibrium models for asset valuation. They inform us which shocks command the largest shock prices at alternative investment horizons.

By imposing a continuous-time Brownian information structure, we localize exposures and establish connections between impulse response functions and shock elasticities. It is of interest to explore information structures that accommodate a more general class of Lévy processes. In previous work, [5], we have initiated this analysis for special cases of jump processes. In defining elasticities we may need to adopt convenient ways to normalize the quantity of risk other than using the standard deviation.

Acknowledgments We would like to thank the referee for useful comments.

Appendix

Derivations for the model with predictable consumption dynamics

Model with persistent components in consumption dynamics

To derive the recursion for the parameters in the conditional expectation (26), guess a solution

$$E\left[M_t \mid X_0 = x\right] = \exp\left[\theta_0(t) + \theta_1(t) \cdot x_1 + \theta_2(t) x_2\right].$$

To derive equations of interest, differentiating the right-hand side with respect to time should agree with the infinitesimal generator applied to the conditional expectation on the left-hand side viewed as a function of x. This relationship implies the following set of ordinary differential equations, each with initial condition $\theta_i(0) = 0$:

$$\frac{d}{dt}\theta_1(t) = \bar{\beta}_1 + (\bar{\mu}_{11})'\theta_1(t)$$
$$\frac{d}{dt}\theta_2(t) = \bar{\beta}_2 + (\bar{\mu}_{12})'\theta_1(t) + \bar{\mu}_{22}\theta_2(t) + \frac{1}{2}\left|\bar{\alpha}' + \theta_1(t)'\bar{\sigma}_1 + \theta_2(t)\bar{\sigma}_2\right|^2$$
$$\frac{d}{dt}\theta_0(t) = \bar{\beta}_0 - (\iota_1)'\left[\bar{\beta}_1 + (\bar{\mu}_{11})'\theta_1(t)\right] - \iota_2\left[\bar{\beta}_2 + (\bar{\mu}_{12})'\theta_1(t) + \bar{\mu}_{22}\theta_2(t)\right]. \quad (32)$$

For the case when X_1 is scalar, we have

$$\theta_1(t) = \frac{\bar{\beta}_1}{\bar{\mu}_{11}}\left(e^{\bar{\mu}_{11}t} - 1\right).$$

Given the solution for $\theta_1(t)$, the ODE for $\theta_2(t)$ is a Riccati equation

$$\frac{d}{dt}\theta_2(t) = q_0(t) + q_1(t)\theta_2(t) + q_2(t)[\theta_2(t)]^2$$

for known parameter functions q_0, q_1 and q_2. Substituting

$$\theta_2(t) = -\frac{1}{q_2(t)u(t)}\frac{d}{dt}u(t)$$

yields a second-order linear differential equation

$$0 = \frac{d^2}{dt^2}u(t) + R_1(t)\frac{d}{dt}u(t) + R_0(t)u(t) \quad (33)$$

with

$$R_1(t) = -\bar{\mu}_{22} - \left(\bar{\alpha}' + \theta_1(t)\sigma_1\right)\sigma_2'$$
$$R_0(t) = \frac{1}{2}|\sigma_2|^2\left(\beta_2 + \bar{\mu}_{12}\theta_1(t) + \frac{1}{2}\left|\bar{\alpha}' + \theta_1(t)\sigma_1\right|^2\right)$$

While this equation does not have a closed form solution, the coefficients $R_1(t)$ and $R_0(t)$ converge to constants R_1^∞ and R_0^∞, respectively, as $t \to \infty$, because $\theta_1(t) \to -\bar{\beta}_1/\bar{\mu}_{11}$. We can therefore characterize the asymptotic behavior of the differential equation (33). The characteristic equation for the local behavior of this ODE as $t \to \infty$ is

$$0 = z^2 + R_1^\infty z + R_0^\infty.$$

The solution to the conditional expectation $E[M_t \mid X_0 = x]$ will then exist only if there is a real solution to this equation.

Value function for recursive utility

We choose a convenient choice for representing continuous values. Similar to the discussion in [23], we use the counterpart to discounted expected logarithmic utility.

$$dV_t = \mu_{v,t}dt + \sigma_{v,t} \cdot dW_t$$

The local evolution satisfies:

$$\mu_{v,t} = \delta V_t - \delta \log C_t - \frac{1-\gamma}{2}|\sigma_{v,t}|^2 \quad (34)$$

When $\gamma = 1$ this collapses to the discounted expected utility recursion.

Let

$$V_t = \log C_t + v(X_t)$$

and guess that

$$v(x) = \bar{v}_0 + \bar{v}_1 \cdot x_1 + \bar{v}_2 x_2$$

We may compute $\mu_{v,t}$ by applying the infinitesimal generator to $\log C + v(X)$. In addition,

$$\sigma_{v,t} = \alpha_c(X_t) + \sigma(X_t)' \frac{\partial}{\partial x} v(X_t).$$

Substituting into (34) leads to a set of algebraic equations

$$\begin{aligned}
\delta \bar{v}_0 &= \bar{\beta}_{c,0} - (\iota_1)' \left(\bar{\beta}_{c,1} + \bar{\mu}_{11}\bar{v}_1\right) - \iota_2 \left(\bar{\beta}_{c,2} + \bar{\mu}_{12}\bar{v}_1 + \bar{\mu}_{22}\bar{v}_2\right) \\
\delta \bar{v}_1 &= \bar{\beta}_{c,1} + \bar{\mu}_{11}' \bar{v}_1 \\
\delta \bar{v}_2 &= \bar{\beta}_{c,2} + \bar{\mu}_{12}' \bar{v}_1 + \bar{\mu}_{22}\bar{v}_2 + \frac{1}{2}(1-\gamma)\left|\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1 + \bar{\sigma}_2' \bar{v}_2\right|^2
\end{aligned}$$

which can be solved for the coefficients $\bar{v}_i$. The third equation is a quadratic equation for $\bar{v}_2$ that has a real solution if and only if

$$\begin{aligned}
D = &\left[\bar{\mu}_{22} - \delta + (1-\gamma)\left(\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1\right)' \bar{\sigma}_2'\right]^2 - \\
&- 2(1-\gamma)|\bar{\sigma}_2|^2 \left(\beta_{c,2} + \mu_{12}' \bar{v}_1 + \frac{1}{2}(1-\gamma)\left|\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1\right|^2\right) \geq 0.
\end{aligned}$$

In particular, the solution will typically not exist for large values of γ. If the solution exists, it is given by

$$\bar{v}_2 = \frac{-\left[\bar{\mu}_{22} - \delta + (1-\gamma)\left(\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1\right)' \bar{\sigma}_2'\right] \pm \sqrt{D}}{(1-\gamma)\left|\bar{\sigma}_2'\right|^2}. \tag{35}$$

We choose the solution with the minus sign and justify the choice in the next subsection.

Stochastic discount factor

The stochastic discount factor has two components: one that is the intertemporal marginal rate of substitution for discounted log utility and the other is a martingale constructed from the continuation value

$$d \log S_t = -\log \delta - d \log C_t + d \log \widetilde{S}_t$$

where $\widetilde{S}$ is a martingale given by

$$d \log \widetilde{S}_t = \sqrt{X_{2,t}}(1-\gamma)\left(\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1 + \bar{\sigma}_2' \bar{v}_2\right)' dW_t - \frac{1}{2} X_{2,t}(1-\gamma)^2 \left|\bar{\alpha}_c + \bar{\sigma}_1' \bar{v}_1 + \bar{\sigma}_2' \bar{v}_2\right|^2 dt.$$

This martingale can be interpreted as a change of measure. The time-0 price of a payoff G_t maturing at time t satisfies

$$E\left[S_t G_t \mid \mathcal{F}_0\right] = \widetilde{E}\left[\exp(-\delta t)\left(\frac{C_t}{C_0}\right)^{-1} G_t \mid \mathcal{F}_0\right].$$

Under the change of measure induced by $\widetilde{S}$, there exists a standard Brownian motion $\widetilde{W}$ such that

$$d\widetilde{W}_t = dW_t - \sqrt{X_{2,t}}\,(1-\gamma)\left(\bar{\alpha}_c + \bar{\sigma}_1'\bar{v}_1 + \bar{\sigma}_2'\bar{v}_2\right)dt.$$

Substituting this relationship for the law of motion of X_2 in (24) yields

$$dX_{2,t} = \widetilde{\mu}_{22}\left(X_{2,t} - \frac{\bar{\mu}_{22}}{\widetilde{\mu}_{22}}\right)dt + \sqrt{X_{2,t}}\bar{\sigma}_2 d\widetilde{W}_t$$

where the mean-reversion coefficient satisfies

$$\widetilde{\mu}_{22} = \bar{\mu}_{22} + (1-\gamma)\,\bar{\sigma}_2\left(\bar{\alpha}_c + \bar{\sigma}_1'\bar{v}_1 + \bar{\sigma}_2'\bar{v}_2\right) = \delta \pm \sqrt{D}.$$

Therefore, the solution for $\bar{v}_2$ in Eq. (35) with the minus sign leads to $\widetilde{\mu}_{22} < 0$, which implies stable dynamics for X_2. Hansen and Scheinkman [15] provide a rigorous general justification of this choice.

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