Preliminary and Incomplete

# Risk and Robustness in General Equilibrium

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## 1. Introduction

A 'robust decision maker' suspects that his model is misspecified and makes decisions that work well over a restricted range of specification errors around a base or 'reference' model. In the context of a discrete time linear-quadratic permanent income model, Hansen, Sargent and Tallarini (1997) (HST) used a single parameter – taken from a version of Jacobson and Whittle's 'risk-sensitivity' correction – to index the class of relevant model misspecifications from which the decision maker seeks protection. In that setting, HST showed how robust decision making would generate behavior that looks like risk aversion. HST calculated how much 'preference for robustness' – interpreted as a form of aversion to a particular kind of Knightian uncertainty – it takes to put 'market prices of risk' into empirically plausible regions.

This paper extends the analysis of HST to economies with equilibrium allocations that solve continuous time optimal resource allocation problems. Most of our results exploit the analytical convenience of continuous-time diffusions. Working in continuous time lets us leave the confines of a linear-quadratic framework: objective functions can be nonquadratic and state evolution equations can be nonlinear. Leaving the linear-quadratic framework makes it easier to link our robustness parameter to commonly used measures of risk aversion both in decision problems and in security market data. It also gives us a general characterization of a precautionary mechanism in decision-making induced by a concern for robustness.

Previously, Epstein and Zin (1989) and Duffie and Epstein (1992) broke the link between intertemporal substitution and risk by introducing risk adjustments in evaluating future utility indices. We demonstrate that concerns about robustness can imitate risk aversion in decision-making in two ways. First we reinterpret one of Epstein-Zin's (1989) recursions as reflecting a preference for robustness rather than aversion to risk. Then we recast the resulting (shadow) market price of risk partly as a 'specification-error' or 'model uncertainty' premium.

We motivate robustness by specifying a 'reference model' and particular nearby models. The decision maker specifies a family of 'candidate' nearby models. Looking at these nearby models is the decision maker's response to his suspicion that the reference model is 'misspecified.' We measure the discrepancy between the reference model and the nearby models by using relative entropy, an expected log likelihood ratio, where the expectation is taken over the distribution from the candidate nearby model. Relative entropy is a key tool in the theory of large deviations, a powerful mathematical theory about rates at which initially unknown distributions can be learned from empirical evidence.

Superficially at least, the perspective of the 'robust' controller differs substantially from that of a 'learner'. In our dynamic settings, the robust decision maker accepts the presence of model misspecification as a permanent state of affairs, and devotes his thoughts to designing robust controls, rather than, say, thinking about recursive ways to use data to improve his model specification over time. But as we shall see, the same formulas that in large deviation theory underpin theories of 'learning' also give representations of value functions appropriate for robust decision making. 'Learning' and 'robust decision' share a need for a convenient way of measuring and interpreting the proximity of two probability distributions, despite the differing orientations – of the robust decision maker to 'plan against the worst', and of the learner to collect enough data to decide. In our settings the robust decision maker is just being sensible in planning against model misspecification that a 'learner' could not confidently dispose of even after many observations.

## 2. Misspecification in Discrete Time

In this section we ignore the choice of the control and study an exogenous state evolution equation. We consider first a Markov process for the state vector in either discrete or continuous time. This process can be modeled by specifying a family (semigroup) of conditional expectation operators:

$$\mathcal{T}_s(\phi)(y) = E\left[\phi(x_s)\,|x_0 = y\right]$$

where the index $s$ ranges over nonnegative numbers when time is continuous and over nonnegative integers when time is discrete. The domain of the conditional expectation operators contains at least the space of bounded, continuous functions. It typically can be expanded to a larger space, but the manner in which it is done depends processes being modeled. Such extensions are germaine to our analysis because we apply the conditional expectation operators to value functions, which for some control problems can be unbounded.

Given a current-period reward function, to compute a value function for a fixed Markov process under discounting, one solves the fixed-point problem:

$$V(x) = sU(x) + \exp(-s\delta)\,\mathcal{T}_s V(x)$$

where $U$ is a current period reward function, $\delta$ is the subjective rate of discount, $s$ is the period length, and $V$ is the corresponding value function. Recursive utility formulations generalize this fixed point problem by replacing $\mathcal{T}_s$ with an alternative transformation of the value function, which we denote $\mathcal{R}_s$. We aim to motivate a particular specification of $\mathcal{R}_s$ incorporating a wish for robustness. Our notion of robustness is represented with a class of conditional expectation operators for nearby perturbations of the original Markov process for the state evolution.

For the remainder of this section, we consider misspecification of a discrete-time process, taking the time increment to be unity. We start with a 'reference process' interpreted as a model that is to be treated only as a 'good approximation' of a possibly more complicated model. People agree on the 'reference model' perhaps based on having observed common historical data. They also worry about a common class of potential misspecifications in the form of perturbations about the reference model. Agreement about the reference model and the potential misspecifications is like imposing 'rational expectations' in our framework.

In this section, the 'reference model' is specified as a discrete-time Markov process. In discrete time it suffices to specify the one-period conditional expectation operator: $\mathcal{T}_1 \equiv \mathcal{T}$. Given the Markov structure, the remaining conditional expectation operators can be constructed via the formula:

$$\mathcal{T}_j = (\mathcal{T})^j$$

for integer $j$. The operator $\mathcal{T}$, or the corresponding Markov transition density, forms our 'reference model.'

To characterize model misspecification, we introduce a family of 'candidate models.' Concern for misspecification of a reference model is reflected by allowing discrepancies between a permissible candidate model and the reference model, and somehow evaluating utility over a collection of candidate models.

### 2.1. Markov Perturbations

We start by introducing a particularly simple family of candidate models and discuss later what happens when we enrich this class. Our simple starting point by design imitates a formalization used in large deviation theory for Markov processes (*e.g.* see Dupuis and Ellis, 1997). We generate a class of candidate models, in the form of a simple alteration of the reference model's Markov process. For a strictly positive function $w$, we form a distorted expectation operator:

$$\mathcal{T}^w(\phi) = \frac{\mathcal{T}(w\phi)}{\mathcal{T}(w)}.$$

The operator $\mathcal{T}^w$ gives rise to an alternative Markov specification for the state evolution.[1] Given this construction, $\frac{w(z)}{\mathcal{T}(w)(y)}$ is the Radon-Nikodym derivative of the altered or 'twisted' transition law with respect to the reference law. Here $z$ is a dummy variable used to index tomorrow's state and $y$ is used to index today's state. Since $\mathcal{T}^w$ is defined formally only for functions of tomorrow's state, it has the obvious extension to functions $\psi(z, y)$ of both tomorrow's state and today's state. We let $E^w(\cdot|y)$ denote this extension.

[1] However, as we will see later, preserving a Markov structure is not necessary.

To embody the idea that the reference model is good, we want to penalize discrepancies between a candidate model and the reference model. This can be done in a variety of ways. We measure discrepancy by 'relative entropy', defined as the expected value of the log-likelihood ratio (the log of the Radon-Nikodym derivative above), where expected value means the conditional expectation evaluated with respect to the density associated with the twisted or candidate model (not the reference model).[2] For a candidate model indexed by $w$, this is

$$\begin{aligned} I(w)(y) &\equiv E^{w}\left[\log \frac{w(z)}{\mathcal{T}(w)(y)}|y\right] \\ &= \mathcal{T}^{w}(\log w)(y) - \log\left[\mathcal{T}(w)(y)\right] \geq 0. \end{aligned}$$

Relative entropy is not a metric because it treats the reference model and the candidate model asymmetrically. This asymmetry emerges because the expectation is evaluated with respect to the 'twisted' or candidate distribution, not the reference one. Relative entropy is prominent in both information theory and large deviation theory, and it is known to satisfy several nice properties. (See Shore and Johnson, 1980 and Csiszar, 1991 for axiomatic justifications.) Not only is $I(w)$ nonnegative, but $I(w) = 0$ if $w$ is constant.[3] Substituting for $\mathcal{T}^{w}$ gives:

$$\begin{aligned} I(w) &= \frac{\mathcal{T}\left[w \log(w)\right]}{\mathcal{T}(w)} - \log\left[\mathcal{T}(w)\right] \\ &= E\left[\frac{w(z)}{E\left[w(z)|y\right]} \log\left(\frac{w(z)}{E\left[w(z)|y\right]}\right)\right]. \end{aligned} \tag{2.1}$$

To summarize, we use relative entropy $I(w)$ to measure how far candidate models (indexed by $w$) are from the reference model. In robust decision-making we pay particular attention to candidate models with small relative entropies.

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[2] For readers of Dupuis and Ellis (Chapter 1, Section 4), think of the transition density associated with $\mathcal{T}$ as Dupuis and Ellis's $\theta$; and think of $\frac{w(z)}{\mathcal{T}(w)(y)}$ as Dupuis-Ellis's Radon-Nikodym derivative $\frac{d\gamma}{d\theta}$. For Dupuis and Ellis, relative entropy is $\int \log\left(\frac{d\gamma}{d\theta}\right) d\gamma$.

[3] For Markov specifications with stochastically singular transitions, $\frac{w(z)}{\mathcal{T}(w)(y)}$ may be one even when $w$ is not constant. For these systems, we have in effect over parameterized the perturbations, although in a harmless way.

## 2.2. A Robust Value-Function Recursion

Let $V$ be a value function and $\theta \leq 0$ a parameter. Consider the following problem:

*Problem A*
$$\inf_{w>0} J(w)$$

where
$$J(w) \equiv -\frac{1}{\theta} I(w) + \mathcal{T}^{w}(V).$$

The first term is a weighted entropy measure and the second is the expected value of the value function using the twisted probability model indexed by $w$. The motivation for this problem is that we consider the expectation of next period's value function given the current period's beliefs indexed by $w$. This is depicted by the second term in the objective function. We penalize departures from the reference model using a relative entropy measure of discrepancy weighted by $\frac{-1}{\theta}$. Decreasing the absolute magnitude of $\theta$ increases the penalty for deviating from the 'reference model.'

A solution to Problem A is:

$$w^{*} = \exp(\theta V),$$

which attains the minimized value

$$J(w^{*}) = \frac{1}{\theta} \log \mathcal{T}[\exp(\theta V)].$$

The solution $w^{*}$ is not unique because a scaled version of this function also obtains the same objective. However, the minimized value of objective is unique as is the associated probability law. The construction of the solution assumes that $\mathcal{T}$ can be evaluated at $\exp(\theta V)$.

To verify that $w^{*}$ is the solution, write:

$$I(w) = I^{*}(w/w^{*}) + \frac{\mathcal{T}(w \log w^{*})}{\mathcal{T}(w)} - \log \mathcal{T}(w^{*})$$

where

$$I^{*}(w) = \frac{\mathcal{T}^{*}(w \log w)}{\mathcal{T}^{*}(w)} - \log \mathcal{T}^{*}(w)$$

and

$$\mathcal{T}^{*}\phi \equiv \frac{\mathcal{T}(w^{*}\phi)}{\mathcal{T}(w^{*})}.$$

Notice that $I^*$ is itself interpretable as a measure of relative entropy and hence $I^*(w/w^*) \geq 0$. Thus the criterion $J$ satisfies the inequality:[4]

$$\begin{aligned} J(w) &= -\frac{1}{\theta}\left[I^*(w/w^*) + \frac{\mathcal{T}(w\log w^*)}{\mathcal{T}(w)} - \log\mathcal{T}(w^*)\right] + \mathcal{T}^w(V) \\ &\geq -\frac{1}{\theta}\left[\frac{\mathcal{T}(w\log w^*)}{\mathcal{T}(w)} - \log\mathcal{T}(w^*)\right] + \mathcal{T}^w(V) \\ &= \frac{1}{\theta}\log\mathcal{T}\left[\exp(\theta V)\right] \\ &= J(w^*). \end{aligned}$$

Thus the solution to this entropy-penalized problem is a risk-sensitive adjustment to the value function where

$$\mathcal{R}(V) = \frac{1}{\theta}\log\{\mathcal{T}\left[\exp(\theta V)\right]\}. \tag{2.2}$$

### 2.3. A Larger Class of Perturbations

While formula (2.1) exploits the specific Markov structure of the candidate model, the relative entropy measure has a straightforward extension to a much richer class of perturbations. For instance, suppose that $w$ is a strictly positive function that is allowed to depend on both $z$ and $y$. In this case we extend measure of entropy to be the following:

$$I(w) = E\left[\frac{w(z,y)}{E\left[w(z,y)|y\right]}\log\left(\frac{w(z,y)}{E\left[w(z,y)|y\right]}\right)\right].$$

The solution to this extended version of Problem A remains the same. In other words, it suffices for the 'control' $w$ to depend only on the state tomorrow. Notice the perturbation just described preserves the first-order Markov structure. That is, the 'twisted' process remains a first-order Markov process.

First-order Markov perturbations are no doubt special. So it is of interest to extend the class of perturbations further. We construct a bigger class of perturbations by starting with positive (and appropriately measurable) functions that depend on the entire past history of the Markov state for the reference process. Let $t+1$ denote tomorrow's date. Form the Radon-Nikodym derivative as:

$$h_{t+1} = \frac{w(z_{t+1}, z_t, z_{t-1}, ...)}{E\left[w(z_{t+1}, z_t, z_{t-1}, ...)|z_t, z_{t-1}, ...\right]}.$$

[4] This proof mimics the proof of Proposition 1.4.2 in Dupuis and Ellis (1997), but is included for completeness.

In other words, the time $t+1$ Radon-Nikodym derivative is a strictly positive random variable that is measurable with respect to the current and past values of the Markov state (for the reference process). This random variable is constrained to have mean one conditioned on $z_t, z_{t-1}, \ldots$. The time $t$ conditional entropy is then:

$$I_t\left(h_{t+1}\right) = E\left[h_{t+1} \log h_{t+1} | z_t, z_{t-1}, \ldots\right]$$

and the time $t$ counterpart to the objective function $J$ for Problem A is:

$$J_t\left(h_{t+1}\right) = -\frac{1}{\theta} I_t\left(h_{t+1}\right) + E\left[h_{t+1} V\left(z_{t+1}\right) | z_t, z_{t-1}, \ldots\right].$$

Given that the process $\{z_t\}$ is Markov under the reference model, the solution to the counterpart to the control problem A is to let the 'control' $h_{t+1}$ be a function of the Markov state $z_{t+1}$ alone. Thus our solution to Problem A extends to this richer class of perturbations, including ones for which the time $t+1$ distortion fails to preserve the Markov structure. The essential feature of these perturbations is that the transition probability distribution for the candidate model be absolutely continuous with respect to the transition probabilities of the reference model; this manifests itself in the restriction that $h_{t+1} > 0$. Absolute continuity makes the log likelihood ratio criterion well defined.

## 3. Misspecification in Continuous Time

We now study the continuous-time counterpart to Problem A. As we will see subsequently, the continuous-time formulation can simplify the analysis. For instance, we will show that when the reference model is a diffusion, a specification of $w$ just introduces a nonzero drift specification into Brownian motion increments that drive the model. Specific versions of Markov jump processes and mixed jump diffusion models can also be handled. In the case of a Markov jump process, the misspecification $w$ alters both the jump intensities and the chain probabilities that dictate jump locations when a jump takes place.

The operator formulation of continuous-time Markov processes entails specifying an infinitesimal generator $\mathcal{A}$. The conditional expectations are then given heuristically by:

$$\mathcal{T}_s = \exp\left(s\mathcal{A}\right).$$

This construction is formalized using a Yosida approximation. The domains of the conditional expectation operators $\mathcal{T}_s$ will contain at least the space $\hat{C}$ of continuous functions $\phi$ with limit zero as $|x| \to \infty$; these are equipped with the sup norm. The family or semigroup of conditional expectation operators $\{\mathcal{T}_s \geq 0\}$ is assumed to be a Feller semigroup, which among other things implies that

$$\lim_{s \downarrow 0} \mathcal{T}_s = \mathcal{I}$$

where $\mathcal{I}$ denotes the identity operator. (See Ethier and Kurtz, 1986, Chapter 1, for a general discussion of semigroups and Chapter 4 for a discussion of Feller semigroups.)

### 3.1. Examples

When the reference model is a diffusion, the generator is a second-order differential operator:

$$\mathcal{A}\phi = \mu \cdot \frac{\partial \phi}{\partial x} + \frac{1}{2}\text{trace}\left(\Sigma \frac{\partial^2 \phi}{\partial x \partial x'}\right),$$

where the coefficient vector $\mu$ is the drift of the diffusion and the coefficient matrix $\Sigma$ is the diffusion coefficient. The corresponding stochastic differential equation is:

$$dx_t = \mu(x_t)\, dt + \Sigma^{1/2}(x_t)\, dB_t$$

where $\{B_t\}$ is a multivariate standard Brownian motion. In this case the generator is not a bounded operator and its domain excludes some of the functions in $\hat{C}$. Nevertheless, the generator's domain includes at least functions that are twice differentiable and have compact support. We have reason to extend this domain in our later analysis.

When the reference model is a Markov jump process, the generator can be represented as:

$$\mathcal{A}\phi = \lambda\left[\mathcal{S}\phi - \phi\right]$$

where the coefficient $\lambda$ is a Poisson intensity parameter that dictates the jump probabilities and $\mathcal{S}$ is a conditional expectation operator that encodes the transition probabilities conditioned on a jump taking place. The intensity parameter can depend on the Markov state and is bounded and nonnegative. In this case the domain of both the generator and the semigroup can be extended to the space of bounded Borel measurable functions, again equipped with the sup norm. The generator is a bounded operator on this enlarged space, and the exponential formula for recovering the conditional expectation operators can be defined formally using the standard series expansion of an exponential function. (See Ethier and Kurtz, 1986, Chapter 4).

### 3.2. Markov Process Perturbations

Next we define a family of perturbations of the 'reference' Markov process. We do so by reproducing the initial set of perturbations we used in discrete time and taking limits. In so doing, we will construct 'perturbed' generators. At the outset we proceed heuristically, partly imitating our discrete-time investigation. Let $w$ be a positive function. For a small time interval $\epsilon$ form:

$$\mathcal{T}_\epsilon^w \phi = \frac{\mathcal{T}_\epsilon (w\phi)}{\mathcal{T}_\epsilon (w)}.$$

To construct a 'twisted' generator of continuous-time process associated with $w$, we compute:

$$\begin{aligned} \mathcal{A}^w \phi &\equiv \lim_{\epsilon \downarrow 0} \frac{\mathcal{T}_\epsilon^w \phi - \phi}{\epsilon} \\ &= \frac{\mathcal{A}(w\phi) - \phi \mathcal{A}(w)}{w}. \end{aligned}$$

The resulting formula is well defined provided that $w$ and $w\phi$ are both in the domain of the generator. Since our candidate generator is obtained as a limit of discrete-time generators, we must verify that the $\mathcal{A}^w$ actually generates a Feller semigroup. One way to do this is to verify the postulates of the Hille-Yosida Theorem (Ethier and Kurtz, 1986, page 165). Instead, in later sections we will explore this construction directly for the 'reference' processes that interest us.

To study entropy penalizations in continuous time, let $I_\epsilon(w)$ be the small increment counterpart to the discrete-time entropy measure $I(w)$:

$$I_\epsilon (w) = \frac{\mathcal{T}_\epsilon [w \log (w)]}{\mathcal{T}_\epsilon (w)} - \log [\mathcal{T}_\epsilon (w)].$$

While $I_\epsilon(w)$ converges to zero as $\epsilon$ declines to zero, its 'derivative' is nondegenerate:

$$I'(w) \equiv \lim_{\epsilon \downarrow 0} \frac{I_\epsilon (w)}{\epsilon} = \frac{\mathcal{A}[w \log (w)]}{w} - \frac{\log w}{w} \mathcal{A}(w) - \frac{1}{w} \mathcal{A}(w).$$

Since $I_\epsilon(w)$ is nonnegative, so is $I'(w) \geq 0$, and $I'(w)$ equals zero when $w$ is constant.

Combining these two limiting operators, we can construct a continuous-time counterpart to criterion $J$:

$$J'(w) = -\frac{1}{\theta} I'(w) + \mathcal{A}^w V.$$

Thus to perform a robust version of continuous-time value function updating we are led to solve:

*Problem B* $$\inf_{w>0, w \in D} J'(w)$$

where $D$ is constructed so that both $w$ and $wV$ are in the domain of the generator.[5]

As in discrete time, we will verify that

$$w^* = \exp(\theta V)$$

solves Problem B. The resulting optimized value of the criterion is:

$$J'(w^*) = \frac{1}{\theta} \frac{\mathcal{A}[\exp(\theta V)]}{\exp(\theta V)}.$$

To check the solution, we imitate the argument we gave for the discrete-time counterpart. We start by changing the reference point for the relative entropy measure by using the 'twisted' generator $\mathcal{A}^* = \mathcal{A}^w$ with $w = w^*$ in place of $\mathcal{A}$:

$$I^{*\prime}(w) \equiv \frac{\mathcal{A}^*[w \log(w)]}{w} - \frac{\log w}{w} \mathcal{A}^*(w) - \frac{1}{w} \mathcal{A}^*(w).$$

It may be verified that

$$\begin{aligned} I^{*\prime}(w/w^*) &= I'(w) - I'(w^*) - \theta \mathcal{A}^w(V) + \theta \mathcal{A}^*(V) \\ &= -\frac{1}{\theta} \left[ J'(w) - J'(w^*) \right]. \end{aligned}$$

Since the left-hand side is always nonnegative and $\theta$ is negative, it follows that $w^*$ solves Problem B.

[5] Given the presence of the inf instead of a min the restrictions on $w$ are not problematic. However, restricting $V$ to be in the domain of generator is sometimes too severe. This restriction can be relaxed by instead using the extended generator.

### 3.3. Risk-Sensitive Recursion

In this subsection, we deduce the continuous-time counterpart to the risk sensitive recursion (2.2). This allows us to find a link in continuous time between entropy-based robustness and recursive utility theory as formulated by Duffie and Epstein (1992). The small interval counterpart to operator $\mathcal{R}$ given in (2.2) is

$$\mathcal{R}_\epsilon V \equiv \frac{1}{\theta} \log \mathcal{T}_\epsilon \left[\exp\left(\theta V\right)\right].$$

As $\epsilon$ declines to zero, this operator collapses to the identity operator. Our interest is in the derivative (with respect to $\epsilon$) of this operator (evaluate at $\epsilon = 0$):

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{\mathcal{R}_\epsilon V - V}{\epsilon} &= \frac{1}{\theta} \frac{\mathcal{A}\left[\exp\left(\theta V\right)\right]}{\exp\left(\theta V\right)} \\ &= J'\left(w^*\right). \end{aligned}$$

We will have more to say about this connection between entropy and risk sensitivity after we add specificity to the continuous-time processes.

### 3.4. A Robust Value Function Recursion

We now revisit the question we posed out the outset of section two. In discrete time, the usual fixed point problem for evaluating a time-invariant control law under discounting and an infinite horizon is:

$$V\left(x\right) = \epsilon U\left(x\right) + \exp\left(-\epsilon\delta\right) \mathcal{T}_\epsilon V\left(x\right)$$

where $U$ is the current-period reward function and $\delta$ is the subjective rate of time discount. We obtain continuous-time counterpart by subtracting $V$ from both sides, dividing by $\epsilon$, and taking limits. This results in:

$$0 = U - \delta V + \mathcal{A}V.$$

When we introduce adjustments for model misspecification, we modify this to be

$$0 = U - \delta V + \frac{1}{\theta} \frac{\mathcal{A}\left[\exp\left(\theta V\right)\right]}{\exp\left(\theta V\right)},$$

which can be viewed as one of the utility recursions studied by Duffie and Epstein (1992).

The connection between risk sensitivity and robustness is familiar from the literature on risk-sensitive control (*e.g.* see James (1992) and Runolfsson (1994)). In contrast to that work, we consider control problems with discounting, which leads to study a recursive formulations of risk sensitivity. The resulting recursion is the continuous-time generalization of one studied by Hansen and Sargent (1995).[6]

## 4. Continuous-Time Diffusions

In this section we study the solution to Problem B when the underlying reference model is a diffusion. We then describe a larger class of perturbations in continuous-time that are consistent with our entropy-based notion of robustness.

### 4.1. Markov Perturbations

We start by revisiting the class of perturbations that we studied earlier, but specialized to the diffusion model. Hence, we take the generator to be:

$$\mathcal{A}\phi = \mu \cdot \frac{\partial \phi}{\partial x} + \frac{1}{2}\text{trace}\left( \Sigma \frac{\partial^2 \phi}{\partial x \partial x'} \right),$$

and, as before, we consider a twisted generator:

$$\mathcal{A}^w \phi = \frac{\mathcal{A}(w\phi) - \phi \mathcal{A}(w)}{w}.$$

Note that by the product rule for first and second derivatives:

$$\mathcal{A}(w\phi) = w\mathcal{A}\phi + \phi\mathcal{A}w + \left(\frac{\partial w}{\partial x}\right)' \Sigma \left(\frac{\partial \phi}{\partial x}\right).$$

Thus the twisted generator can be depicted by:

$$\mathcal{A}^w \phi = \mathcal{A}\phi + u'\Sigma^{1/2}\left(\frac{\partial \phi}{\partial x}\right)$$

where we have defined $u$ to be:

$$u \equiv \Sigma^{1/2} \frac{\partial \log w}{\partial x}.$$

[6] The discrete-time recursion is the same as that used by Weil (1993), although Weil does not demonstrate the connection to risk-sensitive control theory.

Thus this form of model misspecification is equivalent to adding a state dependent drift $u$ to the underlying Brownian motion:

$$dx_t = \mu(x_t)\,dt + \Sigma^{1/2}(x_t)\left[u(x_t)\,dt + dB_t\right]. \tag{4.1}$$

The corresponding continuous-time measure of relative entropy is:

$$\begin{aligned} I'(w) &= \frac{\mathcal{A}\left[w\log(w)\right]}{w} - \frac{\log w}{w}\mathcal{A}(w) - \frac{1}{w}\mathcal{A}(w) \\ &= \frac{1}{2}u'u. \end{aligned}$$

Using our representation for 'twisting' the Markov model and for the resulting measure of relative entropy, we can rewrite Problem B as:

*Problem B'*

$$\min_u \left[-\frac{1}{2\theta}u'u + \mathcal{A}V + u'\Sigma^{1/2}\left(\frac{\partial V}{\partial x}\right)\right]$$

with solution:

$$u^* = \theta\Sigma^{1/2}\left(\frac{\partial V}{\partial x}\right).$$

The robust version of the value function recursion is now:

$$U - \delta V + \mathcal{A}V + \frac{\theta}{2}\left(\tfrac{\partial V}{\partial x}\right)'\Sigma\left(\tfrac{\partial V}{\partial x}\right) = 0. \tag{4.2}$$

Runolfsson (1994) deduced the $\delta = 0$ (ergodic control) counterpart to (4.2) to obtain a "robust" interpretation of risk sensitivity. Partial differential equation (4.2) is also a special case of the equation system that Duffie and Epstein (1992) and Duffie and Lions (1992) analyze for stochastic differential utility. They showed that for diffusion models, the recursive utility generalization introduces a variance multiplier, which can be state dependent. The counterpart to this multiplier in our setup is state independent and equal to the so called risk sensitivity parameter $\theta$. Our interest in this variance multiplier stems from its role in restraining entropy instead of enhancing sensitivity to risk. Nevertheless, the mathematical connection to risk sensitivity and recursive utility will permit us to draw on a set of analytical results developed by others when studying the implications of robustness.

### 4.2. Enlarging the Class of Perturbations

The robust discounted value function recursion (4.2) can be obtained in a somewhat different manner. This alternative formulation will permit us to expand the class of *twisted* models that we can consider. Following James (1992), we alter the time $t$ reward function of the decision maker by augmenting $U(x_t)$ with $-\frac{1}{2\theta}|g_t|^2$. The objective of the robust decision-maker has a nonrecursive representation:

$$\int_0^\infty \exp(-\delta t)\left[U(x_t) - \frac{1}{2\theta}|g_t|^2\right]dt, \tag{4.3}$$

and the state evolution is:

$$dx_t = \mu(x_t)\,dt + \Sigma^{1/2}(x_t)\left[g_t dt + dB_t\right]. \tag{4.4}$$

The 'control' vector process $\{g_t\}$ is suitably adapted to the filtration generated by the underlying Brownian motion vector. It has the interpretation of distorting the Brownian by appending a state dependent drift to it. The magnitude of this distortion is restrained by the relative entropy penalty $-\frac{1}{2\theta}|g_t|^2$. As $\theta$ converges to zero, it becomes optimal to set $g$ to zero.

Given the Markov structure, the solution to this control problem can be represented as a time-invariant function of the state vector $x_t$, which we denote: $g_t = u(x_t)$. Notice that (4.1) and (4.4) coincide when under a time invariant control law. The resulting Bellman equation is identical to (4.2), and hence the optimal 'control' satisfies:

$$g_t = \theta\Sigma^{1/2}\left(\frac{\partial V}{\partial x}\right)(x_t).$$

We can now think of $(1/2)|g_t|^2$ as a time $t$ measure of relative entropy but for a more general family of perturbations. Now a perturbation in the reference diffusion process is obtained by introducing a (progressively measurable) process $\{g_t\}$ and appending a drift $\int_o^t g_s ds$ to the Brownian motion $W_t$. This form of perturbation admittedly still looks very special. From our discussion of discrete-time models we are led to expect a large class of perturbations with finite (relative entropy). However, from Girsanov's Theorem, it is evident that there is little scope for further enlargements. Absolute continuity is a potent restriction for continuous-time diffusions.[7] Also, even these seemingly simple drift distortions still result in a rather rich collection of candidate models.

[7] See, *e.g.*, Karatzas and Shreve (1991, pages 190-196), Durrett, (1996, pages 90-93) and Dupuis and Ellis (1997, pages 123-125).

### 4.3. Adding Controls to the Original State Equation

We now add a control vector to the original state evolution equation:

$$dx_t = \mu(x_t, i_t)\, dt + \Sigma^{1/2}(x_t, i_t)\,(g_t d\,t + dB_t) \tag{4.5}$$

and to the return function: $U(x, i)$ where $i_t$ is a control vector. Consider a time-invariant control of the form $i_t = f(x_t)$. The value function for the risk-sensitive (or the robust) control problem satisfies (4.2), except that now the differential operator $\mathcal{A}$ and the reward function $U$ depend on the control law $f$:

$$U(\cdot, f) - \delta V + \mathcal{A}(f)\, V + \frac{\theta}{2}\left(\tfrac{\partial V}{\partial x}\right)' \Sigma(\cdot, f)\left(\tfrac{\partial V}{\partial x}\right) = 0.$$

As we have seen, for a fixed control law $f$, this equation can emerge from solving a single agent 'robust control problem.' Thus we can solve for the optimal (or robust) control law by solving a two-player Markov game:

$$\max_f \min_u \left[U(x, f) - \delta V(x) + \mathcal{A}(f)\, V(x) + \left(\tfrac{\partial V}{\partial x}(x)\right)' \Sigma^{1/2}(x, f)\, u - \frac{1}{2\theta} u' u\right].$$

## 5. Market Price of Risk

In this section, we study the continuous-time analog to the (shadow) market price of risk. This permits us to study how the market price of risk is enhanced by a concern about robustness. That is, we show us how risk aversion as reflected in security prices can be reinterpreted in part as depicting an aversion to model misspecification or Knightian uncertainty. The result will be a formula analogous to one obtained by Hansen, Sargent and Tallarini (1997) as an approximation in discrete time.

We derive this formula by computing shadow prices for a fictitious social planner. Let $\{P_t\}$ denote a valuation process for an asset that is held over a small increment of time and has no dividend payouts during that interval. (Alternatively, we can think of the dividends as being reinvested). Let $\{M_t\}$ be a process used to depict the implied stochastic discount factors. This process is constructed to satisfy the discrete-time pricing relation:

$$\exp(-s\delta) E_t\{\exp[\theta V(x_{t+s})] P_{t+s}M_{t+s}\} = E_t\{\exp[\theta V(x_{t+s})]\}M_tP_t \tag{5.1}$$

for all increments $s$, where $E_t$ is the time $t$ conditional expectation operator associated with the underlying Brownian motion. Thus the stochastic discount factor between time periods $t$ and $t+s$ is given by the ratio $\exp(-s\delta)M_{t+s}/M_t$.

For convenience, we have defined stochastic discount factors in order that assets are valued in accordance to the twisted Markov process where the 'twisting' coincides that obtained as the solution to the optimal resource allocation problem. The Radon-Nikodym derivative used to depict this twisting:

$$\frac{\exp[\theta V(x_{t+s})]}{E_t\{\exp[\theta V(x_{t+s})]\}}$$

which agrees with our earlier analysis of robustness in decision making. By using the twisted probability measure, the process $\{M_t\}$ is directly interpretable as the marginal utility process for the consumption numeraire. Consequently, what we refer to as a stochastic discount factor is also an intertemporal marginal rate of substitution. The twisting of the probability is induced by the preference for robustness, and our aim is to characterize its contribution to the market price of risk.[8]

For the Markov economies we study, the process $\{M_t\}$ is representable as $M_t = \Phi(x_t)$. For instance, this occurs when there is a single consumption good that is chosen optimally by the social planner to be a time invariant function of the Markov state. Express the local evolution as

$$dM_t = M_t\mu_{m,t}dt + M_t\sigma_{m,t}\cdot dB_t$$

where:

$$\mu_{m,t} = \frac{\partial \log \Phi'}{\partial x}(x_t)\,\mu^*(x_t) + \frac{1}{2M_t}\text{trace}\left(\Sigma^*\frac{\partial^2\Phi}{\partial x\partial x}\right)(x_t),$$

and

$$\sigma_{m,t} = \frac{\partial \log \Phi'}{\partial x}(x_t)\left[\Sigma^*(x_t)\right]^{1/2}.$$

[8] The decomposition we deduce is a bit misleading because a preference for robustness can result in changing the consumption-investment profile and hence have a direct effect on the stochastic discount factor. Instead we are focusing on an additional increment in the market price of risk that will be present even when the equilibrium consumption process is unaltered.

In these formulas $\mu^*(x)$ is the drift coefficient for the Markov process $\{x_t\}$ when the optimal, time-invariant control law for $i_t$ is imposed. Similarly, $\Sigma^*$ is the diffusion matrix associated with the solution to the optimal resource allocation problem.

Suppose the valuation process satisfies:

$$dP_t = P_t \mu_{r,t} dt + P_t \sigma_{r,t} \cdot dB_t.$$

We refer to $\mu_{r,t}$ as the time $t$ instantaneous mean rate of return and $(\sigma_{r,t} \cdot \sigma_{r,t})^{1/2}$ as the corresponding instantaneous standard deviation. In what follows, we will think of modeling an instantaneous return by specifying the pair $(\mu_{r,t}, \sigma_{r,t})$. Equilibrium asset pricing imposes explicit restrictions on this pair, and the instantaneous market price of risk shows depicts the equilibrium trade-off between (local) mean rates of return and the corresponding standard deviations.

As is standard when investigating asset pricing models, we deduce the restrictions on the local mean of an asset return, $\mu_{r,t}$ given the 'loading' vector $\sigma_{r,t}$ on the Brownian motion 'factors'. Thus, we derive the risk prices for the Brownian increments, and then we assimilate these prices to form the market price of risk. Since pricing relation (5.1) holds for all $s$, we find that:

$$-\delta + \mu_{m,t} + \mu_{r,t} + \sigma_{r,t}\sigma'_{m,t} + \theta \left(\sigma_{r,t} + \sigma_{m,t}\right) \left[\Sigma^* \left(x_t\right)\right]^{1/2} \left(\frac{\partial V}{\partial x}\right)(x_t) = 0.$$

This relation is the continuous-time counterpart to the familiar consumption Euler equation. From this relation, it follows that the risk free rate is obtained by setting $\sigma_{r,t}$ to zero:

$$r_{f,t} = \delta - \mu_{m,t} - \sigma_{m,t} \cdot g_t^* \tag{5.2}$$

where $u_t^*$ is robust adjustment to the Brownian motion at date $t$. Thus we may rewrite the instantaneous pricing relation as:

$$\mu_{r,t} - r_{f,t} = -\sigma_{r,t} \cdot \left(\sigma_{m,t} + g_t^*\right). \tag{5.3}$$

Thus the 'risk prices' for the Brownian increments are given by the entries of the vector $-\sigma_{m,t} - g_t^*$. The first term is familiar from the usual investigation of the consumption capital asset pricing model (*e.g.*, Breeden, 1979). The second component $-g_t^*$ is an added premium that is induced by the concern for model misspecification.

The market price of risk is the slope of the mean-standard frontier for the full array of asset returns. Applying the Cauchy-Schwarz inequality to (5.3), it follows directly (5.3) that:

$$|\mu_{r,t} - r_{f,t}| \leq (\sigma_{r,t} \cdot \sigma_{r,t})^{1/2} [(\sigma_{m,t} + g_t^*) \cdot (\sigma_{m,t} + g_t^*)]^{1/2}$$

Consequently, the market price of risk is:

$$mpr_t = \left[\left(\sigma_{m,t} + g_t^{*\prime}\right) \cdot (\sigma_{m,t} + g_t^*)\right]^{1/2}. \tag{5.4}$$

This formula extends an approximation result that Hansen, Sargent and Tallarini (1997) derived for discrete-time economies.[9]

Formulas (5.2) and (5.4) show how a concern for model misspecification alters market-based measures of risk premia.

## 6. A Robust Precautionary Motive

Hansen, Sargent and Tallarini (1997) obtained a robust interpretation of the rational expectations version of the permanent income model of consumption. Under this robustness interpretation, consumers in effect are endowed with a precautionary motive for saving as a device to guard against a family of model misspecifications. In this section, we use a discounted version of a value function expansion due to James (1992) to show that the "robust" mechanism for precautionary behavior carries over to a richer class of control problems. We are also able to distinguish the impact of robustness from the more familiar precautionary motive described by Leland (1968), Kimball (1990) and others. In addition, it permits to reinterpret the precautionary motive investigated by Weil (1993).

The value function expansion of James (1992) relies on a small noise expansion. Instead of investigating a single optimal resource allocation problem, we study a family

[9] Hansen, Sargent and Tallarini (1997) studied permanent income-type economies in which the risk-free rate is pinned down by the specification of the technology. They adjusted the subjective discount factor to compensate for the impact of formula (5.2) in equilibrium.

of problems indexed by the magnitude of the risk. We replace the original evolution equation with:

$$dx_t = \mu\left(x_t, i_t\right) dt + \sqrt{\epsilon}\Sigma^{1/2}\left(x_t, i_t\right) dB_t$$

where $\epsilon$ indexes the magnitude of the risk. Associated with this family of evolution equations is a family of generators: $\mathcal{A}_\epsilon(i)$. As in (4.5), the 'robust' counterpart to this family of evolution equations is:

$$dx_t = \mu\left(x_t, i_t\right) dt + \sqrt{\epsilon}\Sigma^{1/2}\left(x_t, i_t\right)\left(g_t + dB_t\right).$$

Finally, the value functions for the corresponding two-player games satisfy the partial differential equation:

$$U\left(\cdot, i\right) - \delta V_\epsilon + \mathcal{A}_\epsilon\left(i\right) V + \epsilon\frac{\theta}{2}\left(\tfrac{\partial V_\epsilon}{\partial x}\right)' \Sigma\left(\cdot, i\right)\left(\tfrac{\partial V_\epsilon}{\partial x}\right) = 0.$$

To obtain a value function expansion, we differentiate this partial differential equation with respect to $\epsilon$ and evaluate it at $\epsilon = 0$. Of course, the decision rule for the control $i_t$ will also depend on $\epsilon$. Since the control is chosen optimally, this dependence does not alter the first-order (in $\epsilon$) expansion of the value function. Instead we are free to ignore this dependence when differentiating with respect to $\epsilon$.

To obtain a path about which to approximate, we solve the deterministic differential equation:

$$dx_t^0 = \mu\left(x_t^0, i_t^0\right) dt$$

given an initial condition $x_0 = x$ where $i_t^0$ is the time $t$ optimal control in the absence of risk ($\epsilon = 0$. Let $DV$ denote the derivative of the value function with respect to $\epsilon$ evaluated at $\epsilon = 0$. Then this derivative (assuming it exists) satisfies the recursion:

$$-\delta DV + \mu \cdot \left(\tfrac{\partial DV}{\partial x}\right) + W_n + \theta W_r = 0 \tag{6.1}$$

where

$$W_n \equiv \frac{1}{2}\text{trace}\left(\Sigma^0 \tfrac{\partial^2 V_0}{\partial x \partial x'}\right),$$

$$W_r \equiv \frac{1}{2}\left(\tfrac{\partial V_0}{\partial x}\right)' \Sigma_0 \left(\tfrac{\partial V_0}{\partial x}\right),$$

and $\Sigma_0$ is the diffusion matrix when computed at $x_t^0, i_t^0$.

Notice that (6.1) is a first-order partial differential equation in $DV$. Its solution may be depicted as follows. Solve the (deterministic) differential equation:

$$dx_t^0 = \mu\left(x_t^0, i_t^0\right) dt$$

given an initial condition $x_0 = x$ where $i_t^0$ is the optimal control in the absence of uncertainty. Then

$$DV(x) = \int_0^\infty \exp(-\delta t)\left[W_n\left(x_t^0\right) + \theta W_r\left(x_t^0\right)\right] dt.$$

Thus we are led to the following value-function expansion:

$$V(x) \approx V_0(x) + \epsilon DV(x)$$

This expansion is just the discounted counterpart to the one given in Theorem 5.1 of James (1992).

Robustness shows up in this value function expansion through the term $\theta W_r$, and increasing $\theta$ in magnitude enhances the contribution of this term. In the absence of robustness ($\theta = 0$) we obtain the familiar contribution $W_n$. Notice that while $W_n$ depends on the second derivative of the value function from the deterministic control problem, $W_r$ is constructed from the first derivative of this value function. Thus the precautionary motive introduced by robustness has a rather different character than that obtained in the absence of robustness. Moreover, for a given (small) amount of 'risk' (a given $\epsilon$), $\theta \int_0^\infty \exp(-\delta t) W_r(x_t^0) dt$ measures how much utility loss can be attributed to the robustness motive. It may be used as a device for interpreting the robustness parameter 'theta.'

## 7. An Alternative Formulation of Robustness

In our treatment of entropy-based robustness, we have used penalization indexed by the so-called risk sensitivity parameter $\theta$. Suppose that we now alter the problem to mimic instead a recursive formulation of Knightian uncertainty, following in part Epstein and Wang (1994). Instead of penalizing entropy, we constrain each period (instant). In continuous time Markov formulation, we impose the constraint:

$$I'(w) \leq \eta.$$

In what follows, we focus on the special case of a diffusion. Thus we require that the drift distortion satisfy:

$$\frac{1}{2}|g_t|^2 \leq \eta.$$

We find a 'robust' control law by guarding against the worst case appropriately constrained. Again the 'solution' for $g_t$ is a time invariant function of the Markov state, so that $g_t^* = u^*(x_t)$ for some function $u^*$.

The function $u^*$ should solve:

$$\min_u \left[ U - \delta V + \mathcal{A}V + u'\Sigma^{1/2}\left(\frac{\partial V}{\partial x}\right)\right]$$

subject to

$$\frac{1}{2}u'u = \eta.$$

The solution to this problem is:

$$u^* = -\sqrt{2\eta}\frac{\Sigma^{1/2}\left(\frac{\partial V}{\partial x}\right)}{\left[\left(\frac{\partial V}{\partial x}\right)'\Sigma\left(\frac{\partial V}{\partial x}\right)\right]^{1/2}}.$$

Thus the robust version of value function updating now entails solving:

$$U - \delta V + \mathcal{A}V - \sqrt{2\eta}\left[\left(\frac{\partial V}{\partial x}\right)'\Sigma\left(\frac{\partial V}{\partial x}\right)\right]^{1/2} = 0$$

In the small-noise expansions of the penalized "robust control" problem, robustness imitates second-order risk aversion. This is evident because the first term to enter is a variance, not a standard deviation. In the small noise expansion for this alternative 'constrained' robustness the first term to enter will be the standard deviation.[10]

## 8. Multiple Agents

In this section we consider an extension of the model by introducing heterogeneous agents. Our reason for doing so is that familiar efficient allocation rules cease to apply in this framework. Even under optimal resource allocation, individual agents will not be allocated a time-invariant function of the aggregate 'pie.' Instead histories will matter, but in a very particular way. As a consequence, even when allocations are Pareto optimal, multiple agent models can look quite different from single-agent counterparts. Given the close connection between our model of robustness and recursive utility theory, we are able to exploit recent work by Dumas, Uppal and Wang (1997) for justification.

For simplicity, we consider an endowment economy with a single consumption good. The aggregate endowment is given by:

$$c^a = g(x)$$

which is to be allocated across consumers. Abusing notation, we let $U$ denote the common period utility function with arguement $c^j$ for person $j$. Thus we have the resource constraint:

$$\sum_{j=1}^{n} c^j = g \tag{8.1}$$

We follow Negishi (1960), Lucas and Stokey (1984), Epstein (1987) and others by using Pareto factors to construct a social welfare function used to allocate consumption among the individual consumers. This function will be constructed so that the intertemporal marginal rates of substitution are equated. Asset prices are then built from the common marginal rates of substitution.

[10] This difference shows up because in the latter case the 'multiplier' on specification error constraint will shrink to zero when the noise covariance diminishes to zero. On the other hand, we previously held fixed the penalty parameter, which we may think of as being analogous to a multiplier on a specification-error constraint. When taking small noise expansions, it is not clear that the penalty parameter $\theta$ should be held fixed, nor is it clear that $\eta$ should be held fixed.

Dumas, Uppal and Wang (1997) show that efficient allocations may be found by deducing a social welfare function that depends on an endogenous state vector of Pareto factors $\lambda_t$ for each $t$. Each component of the Pareto factor vector is matched to an indivual (or individual type) in the economy. Thus the time $t$ Pareto factor for person $j$ is denoted $\lambda_t^j$. The social welfare function $S$ depends both on the exogenous state vector $x$ and the endogenous state vector $\lambda$. This function is a homogeneous of degree one function that satisfies the partial differential equation:

$$0 = \max_c \sum_{j=1}^{n} \lambda^j \left[ \left(\theta S^j + 1\right) U\left(c^j\right) - \frac{\delta}{\theta}\left(\theta S^j + 1\right) \log\left(\theta S^j + 1\right) + \mathcal{A} S^j \right] \tag{8.2}$$

subject to (8.1) where the $U$ notation is now used to the common instanteous utility function for each consumer and $S^j$ is:

$$S^j(x, \lambda) = \frac{\partial S(x, \lambda)}{\partial \lambda^j}.$$

We continue to use the $\mathcal{A}$ notation to depict the local evolution of $\{x_t\}$ (but not of the Pareto factor vector process $\{\lambda_t\}$):

$$\mathcal{A} S^j = \frac{\partial S^j(x, \lambda)}{\partial x} \cdot \mu(x) + \frac{1}{2} \text{trace}\left(\Sigma \frac{\partial^2 S^j(x, \lambda)}{\partial x \partial x'}\right).$$

From the social welfare function $W$, we may construct the entropy-adjusted value functions for each individual via:

$$V^j = \frac{\log\left(\theta S^j + 1\right)}{\theta}.$$

Moreover, the evolution of the Pareto factors is given by:

$$d\lambda_t^j = -\lambda_t^j \left(\delta - \theta\left[U^j(x, \lambda) - \delta V^j(x, \lambda)\right]\right) dt$$

where $U^j$ is the function $U$ evaluated at the efficient choice of $c^j$ given as a function of the state vectors $x$ and $\lambda$. Notice that the Pareto factors are locally predictable. We let $\gamma$ denote the composite drift term obtained by stacking the individual drifts.[11]

[11] When $U$ is log, it can be shown that the solution to (8.2) implies that the drift for $\lambda^j$ is $-\delta$ for every consumer and that $V^j = V^a + d^j$ where the function $V^a$ is common to all consumers and $d^j$ is constant (across states). Thus the Pareto factors $\rho^j$ share a common constant drift and a common weighted Brownian motion increment. In this log case, standard representative consumer aggregation results apply: individual consumptions are proportional to aggregate consumption with a time and state invariant proportionality factor. See Tallarini (1997) for a quantitative investigation of a single consumer *log* specification using the 'risk-sensitive' utility recursion.

To characterize the alloation rule, we introduce a multiplier $\kappa$ on the resource constraint (8.1). The first-order conditions imply that

$$\rho^j U'\left(c^j\right) = \kappa$$

where

$$\rho^j \equiv \lambda_j \exp\left(\theta V^j\right).$$

Thus the allocations satisfy:

$$c_j = \left(U'\right)^{-1}\left(\frac{\kappa}{\rho^j}\right)$$

where $\kappa$ is determined by (8.1).

In the appendix we show formally how to obtain these formulas from Dumas, Uppal and Wang (1997). By construction, the individual value funtions satisfy the entropy-adjusted partial differential equation:

$$U^j - \delta V^j + \mathcal{A}V^j + \gamma \cdot \frac{\partial V^j}{\partial \lambda} + \frac{\theta}{2}\left(\frac{\partial V^j}{\partial x}\right)' \Sigma \left(\frac{\partial V^j}{\partial x}\right) = 0 \tag{8.3}$$

for $j = 1, 2, ..., n$.

## A. Two Equivalent Ways to Represent Preferences

In this appendix, we deduce two alternative ways to depict prefences for diffusion models. One uses the transformation suggested by Duffie and Epstein (1992) and Duffie and Lions (1992). The other one uses a transformation suggested by Dumas, Uppal and Wang (1997) used to depict the preferences has having endogenous discounting instead of robustness. We consider intially a single-agent formulation, and then we study the multiple-agent counterpart.

### A.1. An Expected-value Recursion

Recall the value-function updating recursion under robustness:

$$U \circ g - \delta V + \mathcal{A}V + \frac{\theta}{2}\left(\frac{\partial V}{\partial x}\right)' \Sigma \left(\frac{\partial V}{\partial x}\right) = 0 \qquad (A.1)$$

where we evaluate the current-period utility function at

$$c = g(x)$$

This equation is nonlinear in the partial derivatives of the value function $V$. Following Duffie and Epstein (1992) and Duffie and Lions (1992), we now transform this equation into one that is linear the partial derivatives but nonlinear in the value function. We accomplish this by taking an increasing transformation of $V$:

$$S = \frac{\exp(\theta V) - 1}{\theta}.$$

Hence $S$ is a monotone transformation of the original value function, and it has a well defined limit when $\theta = 0$ ($S = V$). Inverting this relation, we have

$$V = \frac{\log(\theta S + 1)}{\theta}.$$

Moreover,

$$\exp(\theta V) = \theta S + 1$$

Multiply $(A.1)$ by $\exp(\theta V)$, and we see that

$$(\theta S + 1)\, U \circ g - \frac{\delta}{\theta}(\theta S + 1)\log(\theta S + 1) + \mathcal{A}S = 0. \qquad (A.2)$$

Note that while this partial differential equation is nonlinear in $S$, it is linear in the derivatives of $S$. Duffie and Epstein (1992) depict a corresponding expected value recursion associated with the partial differential equation system.

### A.2. Endogenous Discounting

We deduce a third depiction of the recursion, which is the continuous-time counterpart to the variational utility recursion of Geoffard (1996). This representation has an endogenous discount rate $b_t$ at date $t$ and a corresponding discount factor $\exp(-\int_0^s b_{t+u})du$ between date $t$ and date $t+s$. Eventually, we will introduce a different discount factor process for each consumer, and these discount factors will be relabeled as Pareto factors.

To deduce the discount factor representation of preferences, we follow Dumas, Uppal and Wang (1997) by computing the Legendre transform:

$$U^*(b,c) = \max_S \left[ (\theta S + 1) U(c) - \frac{\delta}{\theta} (\theta S + 1) \log(\theta S + 1) + bS \right]$$

where $S$ is a stand in for the value and $b$ is the stand in for the discount rate. Use the change of variables:

$$S^* = \theta S + 1.$$

Then an equivalent problem to the Legendre transform is:

$$U^*(b,c) = \min_{S^*} \left[ S^* U(c) - \frac{\delta}{\theta} S^* \log(S^*) + \frac{b}{\theta}(S^* - 1) \right].$$

The first-order conditions are:

$$U(c) - \frac{\delta - b}{\theta} - \frac{\delta}{\theta} \log(S^*) = 0.$$

Thus

$$\log(S^*) = \frac{\theta}{\delta} U(c) - \frac{\delta - b}{\delta},$$

and

$$\begin{aligned} U^*(b,c) &= -\frac{b}{\theta} + \frac{\delta}{\theta} S^* \\ &= -\frac{b}{\theta} + \frac{\delta}{\theta} \exp\left[ \frac{\theta U(c) - \delta + b}{\delta} \right]. \end{aligned}$$

Consistent with our discount rate interpretation, we introduce an endogenous state variable $\lambda_t$ with dynamic evolution:

$$d\lambda_t = -b_t \lambda_t dt,$$

We depict preferences by having consumers choose a discount rate process $\{b_t\}$ to minimize:

$$E_0 \left[ \int_0^\infty \lambda_t U^* \left(b_t, c_t\right) \right].$$

Given time invariant Markov dynamics for the optimal resource allocation problem, with consumption being a time invariant function of $x_t$, the minimization problem is Markovian with a time-invariant control law for the discount rate process. Notice that the discount factor process is taken to be locally predictable.

The value function for this problem is given by $\lambda S(x)$ where $S$ solves $(A.2)$. Moreover, the first-order conditions for the discount rate process are obtained by solving:

$$\min_b U^* \left(b, c\right) - bS.$$

Hence $b$ should solve:

$$-\frac{1}{\theta} + \frac{1}{\theta} \exp \left[ \frac{\theta U \left(c\right) - \delta + b}{\delta} \right] - S = 0.$$

Therefore,

$$\theta U \left(c\right) - \delta + b = \delta \log \left(\theta S + 1\right),$$

or

$$b = \delta - \theta \left[ U \left(c\right) - \delta V \left(x\right) \right].$$

The 'endogenous' discount factor rate is pulled away from $\delta$ depending on how close the period utility function is to the entropy adjusted (and discounted) value function.

### A.3. A Multiple-Agent Counterpart

As argued by Dumas, Uppal, and Wang (1997), the single-agent discount factor representation may be used to construct to a multiple agent counterpart in which the discount factor processes of each agent play the role of Pareto factors in an optimal resource allocation problem. Thus we now introduce a Pareto factor process for each person, with evolution given by:

$$d\lambda_t^j = -b_t^j \lambda_t^j dt$$

for $j = 1, 2, ..., n$. We stack the date $t$ $\lambda$'s into a composite (n-dimensional) state vector $\lambda_t$ with (degenerate) Markov dynamics.

For simplicity, we consider an endowment economy in which aggregate consumption is a time invariant function of the Markov state $x_t$:

$$\sum_{j=1}^{n} c^j = g(x). \qquad (A.3)$$

We use the Pareto weight processes to split aggregate consumption across agents. When these processes are proportional, the allocation rules are time-invariant functions of the Markov state. This will be true in the limiting case in which $\theta$ is set to zero (no relative entropy adjustments.) More generally, the efficient allocation rules are history dependent, and the $\lambda$ processes are convenient ways to keep track of this dependence.

We solve the multiple-agent optimal resource allocation problem by computing a social welfare function $S$ that depends on the vector of Pareto factors. This function will be homogeneous of degree one in the vector vector, and individual utilities at each date can be deduced using Euler's decomposition of a homogeneous of degree one function:

$$W(x, \lambda) = \sum_{j=1}^{n} \lambda^j S^j(x, \lambda)$$

where

$$S^j(x, \lambda) \equiv \frac{\partial S(x, \lambda)}{\partial \lambda^j}.$$

Analogous to our single-agent model, we may view $\lambda^j S^j(x, \lambda)$ as the value function for person $j$.

We produce a functional equation for the social value function $S$. This function depends on both the $\lambda$ vector of Pareto weights and the original Markov state vector $x$. When there are multiple consumers, we are led to solve:

$$0 = \max_{c} \min_{b} \sum_{j=1}^{n} \lambda^j \left[ U^* \left( c^j, b^j \right) - b^j S^j \left( x, \lambda \right) + \mathcal{A} S^j \left( x, \lambda \right) \right]$$

subject to the resource constraint

$$\sum_{j=1}^{n} c^j = g\left( x \right).$$

The local operator $\mathcal{A}$ applied to $S^j$ uses derivatives with respect to $x$, but not $\lambda$.

To solve the minimization component of this two-player game, we construct the entropy-adjusted counterpart to be:

$$V^j \equiv \frac{\exp\left( \theta S^j \right) - 1}{\theta}.$$

Then the discount rates for the Pareto factors satisfy:

$$b^j = \delta - \theta \left[ U\left( c^j \right) - \delta V^j \left( x, \lambda \right) \right] \qquad (A.4)$$

for the optimal allocation of consumption across people. Thus the decision about the Pareto factor adjustment is based on a comparison of the current period utility and the current period entropy-adjusted values.

Given the solution for the discount rates $(A.4)$, it can be shown that

$$\exp\left[ \frac{\theta U\left( c^j \right) - \delta + b^j}{\delta} \right] = \exp\left( \theta V^j \right).$$

Therefore, the $S^j$'s also solve heterogeneous agent counterpart to to the Duffie-Lions partial differential equation $(A.1)$:

$$0 = \max_{c} \sum_{j=1}^{n} \lambda^j \left[ \left( \theta S^j + 1 \right) U\left( c^j \right) - \frac{\delta}{\theta} \left( \theta W^j + 1 \right) \log\left( \theta S^j + 1 \right) + \mathcal{A} S^j \right] \qquad (A.5)$$

subject to

$$\sum_{j=1}^{n} c^j = g.$$

## References


Breeden, D. T. (1979), "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics* 7, 265-296.

Csiszar, I. (1991), "Why Least Squares and Maximum Entropy? An Axiomatic approach to Inference for Linear Inverse Problems," *The Annals of Statistics* 19:4, 2032-2066.

Dumas, B. Uppal, R. and T. Wang (1997), "Efficient Intertemporal Allocations with Recursive Utility," manuscript.

Durrett, R., (1996), *Stochastic Calculus: A Practical Introduction*, Boca Raton: CRC Press.

Duffie, D. and L.G. Epstein (1992), "Stochastic Differential Utility," *Econometrica* 60:2, 353-394.

Duffie, D. and P.L. Lions (1992), "PDE Solutions of Stochastic Differential Utility," *Journal of Mathematical Economics* 21, 577-606.

Dupuis, P. and R. S. Ellis (1997), *A Weak Convergence Approach to the Theory of Large Deviations*, New York: John Wiley and Sons.

Epstein, L.G. and T. Wang (1994), "Intertemporal Asset Pricing Under Knightian Uncertainty," *Econometrica* 62:3, 283-322.

Epstein, L.G. and S.E. Zin (1989), "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica* 57:4, 937-969.

Ethier, S.N. and T.G. Kurtz (1986), *Markov Processes: Characterization and Convergence*, NY: John Wiley and Sons.

Fleming, W.H. and W.M. McEneaney (1995), "Risk-Sensitive Control on an Infinite Time Horizon," *SIAM Journal of Control and Optimization* 33:6, 1881-1915.

Geoffard, P. Y. (1996), "Discounting and Optimizing: Capital Accumulation as a Variational Minmax Problem," *Journal of Economic Theory, 66, 53-70*

Hansen, L.P. and T.J. Sargent (1995), "Discounted Linear Exponential Quadratic Gaussian Control," *IEEE Transactions on Automatic Control* 40:5, 968-971.

Hansen, L.P., T.J. Sargent and T.D. Tallarini, Jr. (1997), "Robust Permanent Income and Pricing," manuscript.

James, M.R. (1992), "Asymptotic Analysis of Nonlinear Stochastic Risk-Sensitive Control and Differential Games," *Mathematics of Control, Signals, and Systems* 5, 401-417.

Karatzas, I. and S. E. Shreve (1991), *Brownian Motion and Stochastic Calculus*, Second Edition, New York: Springer-Verlag.

Kimball, M. S. (1990), "Precautionary Savings in the Small and in the Large," *Econometrica* 58, 53-73.

Lucas, R. and N. Stokey (1989), "Optimal Growth with Many Consumers," *Journal of Economic Theory*, 32, 139-171.

Negishi, T. (1960), "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, 12, 92-97

Runolfsson, T. (1994), "The Equivalence between Infinite-horizon Optimal Control of Stochastic Systems with Exponential-of-integral Performance Index and Stochastic Differential Games," *IEEE Transactions on Automatic Control* 39, 1551-1563.

Runolfsson, T. (1995), "Risk-sensitive Control of Stochastic Hybrid Systems," *Proceedings of the 34th Conference on Decision and Control*, New Orleans, LA.

Shore, J.E. and R.W. Johnson (1980), "Axiomatic Derivation of the Principle of Maximum Entropy and the Principle of Minimum-Cross Entropy," *IEEE Transactions on Information Theory* 26:1, 26-37.

Weil, P. (1993), "Precautionary Savings and the Permanent Income Hypothesis," *Review of Economic Studies* 60, 367-383.