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Pricing growth-rate risk

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Abstract We characterize the compensation demanded by investors in equilibrium for incremental exposure to growth-rate risk. Given an underlying Markov diffusion that governs the state variables in the economy, the economic model implies a stochastic discount factor process S. We also consider a reference growth process G that may represent the growth in the payoff of a single asset or of the macroeconomy. Both S and G are modeled conveniently as multiplicative functionals of a multidimensional Brownian motion. We consider the pricing implications of parametrized family of growth processes G^{ϵ}, with $G^0 = G$, as ϵ is made small. This parametrization defines a direction of growth-rate risk exposure that is priced using the stochastic discount factor S. By changing the investment horizon, we trace a *term structure* of risk prices that shows how the valuation of risky cash flows depends on the investment horizon. Using methods of Hansen and Scheinkman (Econometrica 77:177–234, 2009), we characterize the limiting behavior of the risk prices as the investment horizon is made arbitrarily long.

Keywords Markov process · Growth-rate risk · Pricing dynamics · Elasticities

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JEL Classification C52 · G12

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1 Introduction

A standard result from asset pricing theories is the characterization of the local risk-return tradeoff. This tradeoff is particularly simple in the case of Brownian information structures. In mathematical finance, the risk prices are embedded in the transformation to a risk-neutral measure. Applying Girsanov's theorem, this change of measure adds a drift vector to the multivariate standard Brownian motion. The vector of local risk prices is the negative of the drift vector used in constructing the risk-neutral transformation. This price vector reflects the local compensation in terms of the drift for exposure to alternative components of the Brownian motion. With these local prices, we price exposure to linear combinations of the Brownian risks by forming the corresponding linear combination of prices.

While derivative claims are often priced using the risk-neutral measure, structural models of asset prices interpret these prices in terms of the fundamentals of the underlying economy. In this paper, as in [7] and [5], we characterize the compensation demanded by investors for added risk at different time horizons, that is, a *term structure* of risk prices. This compensation will typically depend on how investors discount risky payoffs and the risk they already face. Our approach is as follows. There is an underlying Markov diffusion X that governs the state variables in the economy. The economic model implies a stochastic discount factor process S. We also consider a reference growth process G that may represent the growth in the payoff of a single asset or of the macroeconomy. Both S and G are modeled conveniently as multiplicative functionals of a multidimensional Brownian motion. To feature the role of price dynamics, we normalize the reference growth functional to be a martingale. More generally, this martingale can be the martingale component in a factorization of the growth functional (as in [7]). We consider a parameterized family of growth processes G^ϵ, with $G^0 = G$, and study its pricing implications for payoffs at different horizons. We define the price of growth-rate risk as

$$\rho_t = -\frac{d}{d\epsilon}\frac{1}{t}\log E\big[G_t^\epsilon S_t \mid X_0 = x\big]\Big|_{\epsilon=0}.$$

It is the elasticity of the expected rate of return (per unit of time) with respect to the exposure to growth-rate risk. The expected return implicit in this calculation is the reciprocal of the price $E[G_t^\epsilon S_t \mid X_0 = x]$ since G_t^ϵ has expectation one by construction.

The resulting prices of growth-rate risk extend the local prices to arbitrary investment horizons. While we focus on scalar parameterizations, we can interpret our calculations as producing prices for an arbitrary linear combination of exposure to the Brownian motion risks. By changing the exposure weights, we feature alternative components of the Brownian increments, and thus construct the counterpart to the local risk-price vector.

For a given investment horizon, we characterize our risk prices by applying tools that are used to compute sensitivities of option prices (the "Greeks"). The prices we compute reveal the local risk prices as the horizon t shrinks to zero, i.e.,

$$\lim_{t\downarrow 0}\rho_t = \rho_0.$$

We add to this a characterization of the limit prices as the investment horizon tends to ∞, i.e.,

$$\lim_{t\uparrow\infty}\rho_t=\rho_\infty,$$

along with formulas for the intermediate investment horizons.

2 Mathematical setup

The underlying state vector process X is n-dimensional and satisfies

$$dX_t=\beta(X_t)\,dt+\alpha(X_t)\,dW_t, \tag{2.1}$$

where W is an n-dimensional Brownian motion on a probability space $\{\Omega,\mathcal{F},\mathsf{Pr}\}$ and $\alpha(\cdot)$ is nonsingular. We write $\{\mathcal{F}_t:t\geq 0\}$ for the (completed) Brownian filtration. Assuming that β and α are locally Lipschitz, there exists a unique X that solves (2.1) when $X_0=x$. In this section, we think of $X_0=x$ as fixed or known, but construct assumptions and results that apply to all initializations. In Sect. 6, we shall introduce explicit randomness in X_0 and augment the filtration $\{\mathcal{F}_t:t\geq 0\}$. The "unconditional" expectations of this section will become expectations conditioned on $X_0=x$ in Sect. 5. Moreover, the resulting dependence on x will be of central interest in applications. We use multiplicative functionals M of the form

$$M_t=\exp\biggl(\int_0^t\delta(X_u)\,du+\int_0^t\gamma(X_u)\,dW_u\biggr), \tag{2.2}$$

where

$$\int_0^t\bigl|\delta(X_u)\bigr|\,du<\infty,\qquad \int_0^t\bigl|\gamma(X_u)\bigr|^2\,du<\infty$$

for all t with probability one.[1] The multiplicative functional M given by (2.2) is referred to as parameterized by (δ,γ). Consider two multiplicative functionals, G parameterized by (δ_g,γ_g) and S parameterized by (δ_s,γ_s). The process G captures stochastic growth and the process S stochastic discounting. Both G and S depend on x, but since we only consider a fixed initial condition x, unless there is ambiguity, we omit in the notation the dependence on x.

Asset valuation over a horizon t is represented as

$$E[S_tG_t],$$

where G_t is the asset payoff that is priced. There are two channels that dictate the term structure of risk premia and the associated prices: stochastic discounting and stochastic growth. Our aim is to focus on the latter channel.

[1] As in, e.g. [9], we allow for multiplicative functionals that do not have bounded variation.

Reference [7] (Corollary 6.1) establishes a multiplicative factorization of G as

$$G_t = \exp(\eta t) G_t^o \frac{f(X_0)}{f(X_t)}$$

where G^o is a multiplicative martingale.[2] The exponential growth term $\exp(\eta t)$ is of no consequence for risk prices and can be omitted. Since predictability in S and G alter the term structure of risk premia, one possibility is to feature the role of pricing dynamics by focusing exclusively on the martingale component G^o and constructing perturbations that preserve the martingale property. In what follows, we shall adopt this perspective where $G = G^o$, and hence is restricted to be a martingale. In this case,

$$-\log E[S_t G_t]$$

is the logarithm of the expected return associated with the martingale payoff G_t.

To construct risk prices for any given payoff horizon, we parameterize a family of growth functionals as G^ϵ with $G = G^o$, where G^ϵ is a martingale for each ϵ. The parameterized martingale is constructed to feature exposures to specific combinations of shocks. By altering the parameterization, we explore sensitivity to alternative shocks, thereby constructing counterparts to local risk prices.

As an alternative, we could work with a macro growth functional G that is not necessarily a martingale, but explore parameterized perturbations that are martingales. Then the logarithm of the expected return associated with G_t is

$$\log E[G_t] - \log E[S_t G_t].$$

For this formulation, our baseline martingale G^o is identically one and the counterpart to S for our analysis is either SG or G. With these changes, our forthcoming analysis will continue to be applicable.

Recall that the stochastic exponential of a semimartingale N is a semimartingale $\mathcal{E}(N)$ that solves $\mathcal{E}(N)_t = 1 + \int_0^t \mathcal{E}(N)_{s-}\, dNs$. Since in our case sample paths are continuous,

$$\mathcal{E}(N) = \exp\left(N - \frac{1}{2}[N, N]\right).$$

We assume that the positive martingale G is the stochastic exponential $\mathcal{E}(Z^o)$ of a martingale $Z_t^o = \int_0^t \gamma_g(X_u)\, dW_u$. Consider a family of perturbations G^ϵ of the form

$$G^\epsilon = \mathcal{E}\big(Z^o + \epsilon Z\big),$$

$\epsilon \in (-1, 1)$, where $Z_t = \int_0^t \gamma_d(X_u)\, dW_u$. For the stochastic integrals to be well behaved, $\int_0^t |\gamma_g(X_u)|^2\, du < \infty$ and $\int_0^t |\gamma_d(X_u)|^2\, du < \infty$ with probability one.

The process Z used to construct the perturbation can feature any of the individual components of the underlying Brownian motion. The resulting parameterized family

[2] Strictly speaking, this corollary produces a local martingale rather than a martingale.

expressed in logarithms is

$$\log G_t^\epsilon = \int_0^t \gamma_g(X_u)\,dW_u + \epsilon \int_0^t \gamma_d(X_u)\,dW_u - \frac{1}{2}\int_0^t \left|\gamma_g(X_u) + \epsilon\gamma_d(X_u)\right|^2 du.$$

In this specification, $\epsilon \int_0^t \gamma_d(X_u)\,dW_u$ captures the (growth-rate) risk exposure. By changing γ_d, we alter which Brownian increments are featured in the pricing.

3 Finite-horizon prices

In this section, we apply an approach developed by [2, 3] (see also [1]) to show that

$$\rho_t = -\frac{E[S_t G_t(\int_0^t \gamma_d(X_u)\,dW_u - \int_0^t \gamma_d(X_u)\cdot\gamma_g(X_u)\,du)]}{t E[S_t G_t]}. \tag{3.1}$$

We start by using the multiplicative martingale G to change measure. Then Girsanov's theorem guarantees that $\frac{G^\epsilon}{G} = \mathcal{E}(\epsilon\tilde{Z})$, and $\tilde{Z}_t = \int_0^t \gamma_d(X_u)\,d\tilde{W}_u$, where $\tilde{W}_u = -\int_0^u \gamma_g(X_v)\,dv + W_u$, and $\tilde{W}$ is a Brownian motion on $[0, t]$ under the changed measure $\tilde{\mathsf{Pr}}$. Notice that the functional form of G guarantees that X remains Markov under $\tilde{\mathsf{Pr}}$. We write $\tilde{E}$ for the associated expectations operator. Hence,

$$\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} = \int_0^t \frac{G_u^\epsilon}{G_u}\,d\tilde{Z}_u = \int_0^t \frac{G_u^\epsilon}{G_u}\gamma_d(X_u)\,d\tilde{W}_u. \tag{3.2}$$

If we can interchange on the right-hand side the limit as $\epsilon \to 0$ with the integral sign, then

$$\frac{\frac{G_t^\epsilon}{G_t} - 1}{\epsilon} \longrightarrow \int_0^t \gamma_d(X_u)\,d\tilde{W}_u$$

For each initial value x of X, we may write the price of an asset as a function of the perturbation on the growth factor as

$$U(\epsilon) = E\big[G_t^\epsilon S_t\big] = \tilde{E}\left[\frac{G_t^\epsilon}{G_t} S_t\right].$$

Hence,

$$U'(0) = \lim_{\epsilon\to 0} \frac{\tilde{E}[(\frac{G_t^\epsilon}{G_t} - 1)S_t]}{\epsilon} = \lim_{\epsilon\to 0} \tilde{E}\left[S_t \int_0^t \frac{G_u^\epsilon}{G_u}\gamma_d(X_u)\,d\tilde{W}_u\right].$$

Next, we impose two assumptions that are sufficient for the main result in this section. After establishing this result, we provide sufficient conditions for the second of these assumptions.

Assumption 3.1 *For each* x, $E[(S_t)^2 G_t] < \infty$.

Imposing this restriction is equivalent to assuming that S_t has a finite conditional second moment (in the $\tilde{\mathsf{Pr}}$ measure).

Assumption 3.2 *For each* x,

$$\frac{\frac{G_t^\epsilon}{G_t}-1}{\epsilon} \longrightarrow \int_0^t \gamma_d(X_u)\, d\tilde{W}_u$$

in $L^2(\tilde{\mathsf{Pr}})$ *as* $\epsilon \to 0$.

Proposition 3.3 *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then for each* x,

$$\begin{aligned} U'(0) &= \tilde{E}\left[S_t \int_0^t \gamma_d(X_u)\, d\tilde{W}_u\right] \\ &= E\left[S_t G_t \left(\int_0^t \gamma_d(X_u)\, dW_u - \int_0^t \gamma_d(X_u)\cdot\gamma_g(X_u)\, du\right)\right]. \end{aligned}$$

Proof This follows directly from Hölder's inequality. □

The elasticity of interest is the ratio $-\frac{U'(0)}{tU(0)}$ and is given by (3.1).

We now provide sufficient conditions for Assumption 3.2. To ensure that the process $\frac{G^\epsilon}{G} = \mathcal{E}(\epsilon \tilde{Z})$ is a martingale, we assume Novikov's condition, i.e.,

Assumption 3.4 *For each* x,

$$\tilde{E}\left[\exp\left(\frac{1}{2}\int_0^t \left|\gamma_d(X_u)\right|^2 du\right)\right] < \infty.$$

For fixed $1 \le m < \infty$ and $t > 0$, consider the space L^m of adapted stochastic processes $Y = \{Y_u\}_{0\le u\le t}$, with norm $\|Y\| = (\tilde{E}[\int_0^t |Y_u(\omega)|^m du])^{1/m}$. Notice that $\frac{G^\epsilon}{G}$ converges to 1 almost surely as $\epsilon \to 0$. Another form of convergence is established in the following lemma.

Lemma 3.5 *Suppose Assumption 3.4 is satisfied. Then for each* x, $\lim_{\epsilon\to 0} \frac{G^\epsilon}{G} = 1$ *in* L^m *for any* $m > 1$.

Proof Since $\frac{G^\epsilon}{G} \to 1$ a.s. and convergence a.s. plus convergence in norm implies the convergence in L^m, it suffices to show that for ϵ small, $\frac{G^\epsilon}{G} \in L^m$ and $\|\frac{G^\epsilon}{G}\| \to 1$. Given $m > 1$, let $c_m = \frac{m}{2}(\sqrt{m} + \sqrt{m-1})^2$. If $\epsilon^2 < \frac{1}{2c_m}$, then for each $0 < u \le t$,

$$\tilde{E}\left[\exp\left(c_m \int_0^u \left|\epsilon\gamma_d(X_\tau)\right|^2 d\tau\right)\right] \le \tilde{E}\left[\exp\left(\frac{1}{2}\int_0^t \left|\gamma_d(X_\tau)\right|^2 d\tau\right)\right] < \infty.$$

Jensen's inequality and Theorem 1 of [4] guarantee that for $0 \le u \le t$,

$$1 \le \tilde{E}\left[\left(\frac{G_u^\epsilon}{G_u}\right)^m\right] \le \tilde{E}\left[\exp\left(c_m \int_0^t \left|\epsilon\gamma_d(X_\tau)\right|^2 d\tau\right)\right] < \infty.$$

Monotone convergence assures that

$$\lim_{\epsilon\downarrow 0}\tilde{E}\left[\exp\left(c_m\int_0^t\left|\epsilon\gamma_d(X_\tau)\right|^2 d\tau\right)\right]=1,$$

and thus $\tilde{E}[(\frac{G_u^\epsilon}{G_u})^m]\to 1$, uniformly in $u\le t$. Hence,

$$\left\|\frac{G^\epsilon}{G}\right\|=\left(\tilde{E}\left[\int_0^t\left(\frac{G_u^\epsilon}{G_u}\right)^m du\right]\right)^{1/m}\to 1.$$

□

To control the term in $\gamma_d(X_t)$, we need to impose

Assumption 3.6 *For each x, there exists a constant Γ (which may depend on x) such that*

$$\tilde{E}\left[\left|\gamma_d(X_u)\right|^4\right]\le\Gamma$$

for $0<u\le t$.

Lemma 3.7 *Suppose Assumptions 3.4 and 3.6 are satisfied. Then Assumption 3.2 holds.*

Proof Use (3.2) to represent

$$\frac{\frac{G_t^\epsilon}{G_t}-1}{\epsilon}=\int_0^t\frac{G_u^\epsilon}{G_u}\gamma_d(X_u)\,d\tilde{W}_u.$$

Thus, we must show that $\int_0^t(\frac{G_u^\epsilon}{G_u}-1)\gamma_d(X_u)\,d\tilde{W}_u$ converges in mean-square to zero. Notice that the stochastic integral $\int_0^t(\frac{G_u^\epsilon}{G_u}-1)\gamma_d(X_u)\,d\tilde{W}_u$ has second moment

$$\tilde{E}\left[\int_0^t\left(\frac{G_u^\epsilon}{G_u}-1\right)^2\left|\gamma_d(X_u)\right|^2 du\right]. \tag{3.3}$$

As $\epsilon\to 0$, expression (3.3) converges to zero from the assumptions, Lemma 3.5 for $m=4$ and Hölder's inequality. □

There are many alternative more primitive assumptions that are sufficient for Assumption 3.6. Here, we present two conditions that together imply Assumption 3.6. The first is a slight strengthening of Novikov's condition for G.

Assumption 3.8 *For each x, there exists $c>\frac{1}{2}$ such that*

$$E\left[\exp\left(c\int_0^t\left|\gamma_g(X_u)\right|^2 du\right)\right]<\infty.$$

It is a consequence of Assumption 3.8 that there exists a $p > 1$ such that for $u \le t$,[3]

$$E\big[(G_u)^p \mid X_0 = x\big] \le E\left[\exp\left(c \int_0^t \left|\gamma_g(X_u)\right|^2 du\right)\right].$$

The next assumption guarantees that for each initial value x, there exists a $\Gamma'(x)$ such that for q satisfying $\frac{1}{q} + \frac{1}{p} = 1$, $u \le t$,

$$E\big[\left|\gamma_d(X_u)\right|^q\big] \le \Gamma'(x).$$

Hölder's inequality then assures that Assumption 3.6 holds.

Assumption 3.9

(a) *The functions* $|\gamma_d(x)|$ *are bounded by a polynomial in* $|x|$.
(b) *The coefficients* β *and* α *of* (2.1) *that defines the evolution of* X *satisfy a sublinear growth condition*,

$$\left|\beta(x)\right|^2 + \left|\alpha(x)\right|^2 \le K\big(1 + |x|^2\big),$$

for some constant K.

When Assumption 3.9(b) holds, for each $m \ge 1$ there exists a $C = C(d, K, T, m)$ such that $E[\max_{u \le t} |X_t|^{2m} \mid X_0 = x] \le C(1 + |x|^{2m})e^{Ct}$, if $t \le T$. (A more general result than this is problem 3.15 in [10]). Part (a) of Assumption 3.9 then implies that for each x, there exists a $\Gamma'(x)$ such that for $0 \le u \le t$, $E[|\gamma_d(X_u)|^q] \le \Gamma'(x)$.

4 Short-term limits

We now use the formula

$$\rho_t = \frac{E[S_t G_t (\int_0^t \gamma_d(X_u) \cdot \gamma_g(X_u)\, du - \int_0^t \gamma_d(X_u)\, dW_u)]}{t E[S_t G_t]}$$

to study valuation over short time intervals. Formally we calculate short-horizon limits by computing the drift of an Itô process.

Recall that SG has continuous sample paths that converge to one as t declines to zero. We add to this the following assumption.

Assumption 4.1 *For every* x, $\lim_{t \downarrow 0} E[S_t G_t] = 1$.

This assumption follows from the dominated convergence theorem, provided that we can dominate SG uniformly for small t.

[3] See Theorem 1 in [4].

With this restriction, we are led to compute

$$\rho(x)=\lim_{t\downarrow 0}\frac{1}{t}E\left[S_tG_t\left(\int_0^t\gamma_d(X_u)\cdot\gamma_g(X_u)\,du-\int_0^t\gamma_d(X_u)\,dW_u\right)\right].$$

We calculate this limit as the drift of the Itô process

$$S_tG_t\left(\int_0^t\gamma_d(X_u)\cdot\gamma_g(X_u)\,du-\int_0^t\gamma_d(X_u)\,dW_u\right)$$

at $t=0$. Hence,

$$\rho_0=\gamma_d(x)\cdot\gamma_g(x)-\gamma_d(x)\cdot\left[\gamma_g(x)+\gamma_s(x)\right],$$

and the following proposition holds.

Proposition 4.2 *Suppose Assumption 4.1 is satisfied. Then*

$$\rho_0=-\gamma_s(x)\cdot\gamma_d(x).$$

As we vary the risk exposure vector γ_d, we trace out the local risk prices. This results in the interpretation of $-\gamma_s$ as a vector of local risk prices.[4] As is well known, the local risk price vector is the risk exposure of the stochastic discount factor S. The risk exposure of the stochastic growth process plays no role in this calculation.

5 An integral representation

In this section, we justify the integral representation

$$\rho_t=-\frac{\hat{E}[\hat{e}(X_t)\int_0^t\gamma_d(X_u)\cdot[\kappa(X_u)+\phi(X_u,t-u)]\,du]}{t\hat{E}[\hat{e}(X_t)]} \tag{5.1}$$

under a particular change of measure. We shall describe formally the construction of the distorted probability distribution and the functions κ and ϕ.

Following [7], use the factorization

$$S_tG_t=\exp(\delta t)\hat{M}_t\frac{e(x)}{e(X_t)},$$

where $\hat{M}$ is a multiplicative martingale and (δ,e), solve a principal eigenvalue problem: find $e>0$ such that

$$E\left[S_tG_te(X_t)\right]=\exp(\delta t)e(X_0). \tag{5.2}$$

[4] In general, this limit is computed as in Itô's lemma by using stopping times. When the associated local martingale is in fact a square-integrable martingale, stopping times can be dispensed with.

Since e solves a principal eigenvalue problem, it is smooth and Itô's lemma can be used to show that $\hat{M}$ is a multiplicative process of the form defined in (2.2) above. Write the volatility exposure (the coefficient of dW_t) of $\log \hat{M}$ as $\kappa + \gamma_g$. Change measure using the martingale $\hat{M}$ and express the finite t derivative of interest as

$$\begin{aligned}\rho_t &= \frac{E[S_t G_t(\int_0^t \gamma_d(X_u)\cdot\gamma_g(X_u)\,du - \int_0^t \gamma_d(X_u)\,dW_u)]}{tE[S_t G_t]}\\ &= -\frac{\hat{E}[\frac{1}{e(X_t)}(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du + \int_0^t \gamma_d(X_u)\,d\hat{W}_u)]}{t\hat{E}[\frac{1}{e(X_t)}]}.\end{aligned}$$

Under the $\hat{\cdot}$ change of measure,

$$dW_u = \big[\kappa(X_u) + \gamma_g(X_u)\big]\,du + d\hat{W}_u,$$

where $\hat{W}$ is an n-dimensional Brownian motion (with respect to the filtration generated by the past values of W) and X solves

$$dX_u = \hat{\beta}(X_u)\,du + \alpha(X_u)\,d\hat{W}_u,$$

with $X_0 = x$ and

$$\hat{\beta} = \beta + \alpha(\kappa + \gamma_g).$$

Write $\hat{e} = \frac{1}{e}$. Let $\hat{\mathcal{F}}_u$, $u \geq 0$, denote the (completed) Brownian filtration associated with $\hat{W}$ and note that $e(X_t)$ is measurable with respect to $\hat{\mathcal{F}}_t$. For each t, $u \leq t$, let $D_u\hat{e}(X_t)$ denote the Malliavin derivative of the random variable $\hat{e}(X_t)$. Sufficient conditions for the existence of the Malliavin derivatives of $\hat{e}(X_t)$ are as follows. If the functions $\hat{\beta}$ and α are smooth and have bounded derivatives, then the random variable X_t is in the domain of the Malliavin derivative. In fact, let Y be the first variation process associated to X, that is, $Y_0 = I_n$ and

$$dY_u = \partial\hat{\beta}(X_u)Y_u\,du + \sum_i \partial\alpha_i(X_u)Y_u\,d\hat{W}_u^i. \tag{5.3}$$

Here, ∂F denotes the Jacobian matrix of an $\mathbb{R}^n$-valued function F and α_i is the ith column of the matrix α. Then, for $u \leq t$, we have the $n \times n$-matrix

$$D_u X_t = Y_t Y_u^{-1}\alpha(X_u). \tag{5.4}$$

In addition, if $\hat{e}$ has bounded first derivatives, then

$$D_u\hat{e}(X_t) = \nabla\hat{e}(X_t)\cdot D_u X_t. \tag{5.5}$$

Then the Haussmann–Clark–Ocone formula guarantees[5] that

$$\hat{e}(X_t) = \int_0^t \hat{E}\big[D_u\hat{e}(X_t)\,\big|\,\hat{\mathcal{F}}_u\big]\cdot d\hat{W}_u + \hat{E}\big[\hat{e}(X_t)\big].$$

[5] For a statement of results concerning the Malliavin derivative of functions of a Markov diffusion and the Hausmann–Clark–Ocone formula, see for instance [2], pp. 395 and 396.

Furthermore, (5.3)–(5.5) imply directly that $\hat{E}[D_u\hat{e}(X_t) \mid \hat{\mathcal{F}}_u] = \hat{E}[D_u\hat{e}(X_t) \mid X_u]$.

Define

$$\phi(y, t-u) = \frac{\hat{E}[D_u\hat{e}(X_t) \mid X_u = y]}{\hat{E}[\hat{e}(X_t) \mid X_u = y]}.$$

Then, if we assume the necessary integrability conditions to apply Fubini's theorem,

$$\begin{aligned}
\hat{E}\bigg[\hat{e}(X_t)\int_0^t \gamma_d(X_u)\,d\hat{W}_u\bigg] &= \hat{E}\bigg[\int_0^t \hat{E}\big[\hat{e}(X_t) \bigm| X_u\big]\phi(X_u, t-u)\cdot\gamma_d(X_u)\,du\bigg] \\
&= \hat{E}\bigg[\int_0^t \hat{E}\big[\hat{e}(X_t)\phi(X_u, t-u)\cdot\gamma_d(X_u) \bigm| X_u\big]\,du\bigg] \\
&= \int_0^t \hat{E}\big[\hat{e}(X_t)\phi(X_u, t-u)\cdot\gamma_d(X_u)\big]\,du \\
&= \hat{E}\bigg[\hat{e}(X_t)\int_0^t \phi(X_u, t-u)\cdot\gamma_d(X_u)\,du\bigg].
\end{aligned}$$

Therefore, (5.1) is justified.

Notice that we have an integral decomposition of ρ_t with key ingredient

$$-\big[\kappa(X_u) + \phi(X_u, t-u)\big].$$

Now hold $u = 0$ fixed and depict, as a function of t,

$$-\kappa(x) + \phi(x, t).$$

At $t = 0$, we obtain the local risk price. More generally, we trace out the impact of the shock exposure over the next instant on the price elasticity of the constructed payoff process for alternative investment horizons indexed by t.

6 Long-term limits

In this section, we establish the limiting behavior

$$\lim_{t\to\infty} \rho_t(x) = -\int \gamma_d \cdot \kappa\, d\hat{Q}$$

for some probability measure $\hat{Q}$. In what follows, we show how to construct this measure and justify the limiting behavior.

As a precursor to studying the long-horizon behavior of ρ_t, it is convenient to alter the specification of the Markov process by choosing a probability distribution for the initial state X_0 other than the degenerate construction $X_0 = x$. Since our previous analysis applies for each x, we have some flexibility as to how we do this.

For simplicity, we choose $\Omega = \mathbb{R}^n \times C_0([0,\infty), \mathbb{R}^n)$ with the first coordinate corresponding to the initial condition X_0 and the second coordinate to a realization of Brownian motion. The random vector X_0 is independent of the Brownian motion. Let

$(\mathcal{F}_t^*)$ be the augmented filtration generated by X_0 and W. Since α is nonsingular, this coincides with the augmented filtration generated by X.

We again use the decomposition of [7] to write

$$S_t G_t = \exp(\delta t)\hat{M}_t \frac{e(X_0)}{e(X_t)},$$

where $\hat{M}$ is a multiplicative martingale and (δ, e), solve a principal eigenvalue problem: find $e > 0$ such that

$$E\big[S_t G_t e(X_t) \mid X_0 = x\big] = \exp(\delta t) e(x),$$

which is the same as (5.2) except that we explicitly noted the conditioning. Given a decomposition in this form, we use $\hat{M}$ to change the transition probabilities from date zero to all other dates. Since $\hat{M}$ is a multiplicative martingale with unit expectation (conditioned on X_0), this change of measure preserves the Markov structure and it depends on $X_0 = x$. We still have the freedom to assign an initial probability to X_0, and we shall do so in a convenient manner so as to make the process X stationary under the change of measure.

Associated with the multiplicative functional is a generator $\hat{\mathcal{A}}$ that is an extension of the second-order differential operator

$$\hat{\beta}(x) \cdot \frac{\partial f(x)}{\partial x} + \frac{1}{2}\text{trace}\left[\alpha(x)\alpha(x)^\top \frac{\partial^2 f(x)}{\partial x\, \partial x'}\right]$$

for functions f that are twice continuously differentiable and have compact support on the interior of the state space. As remarked in the previous section, our use of $\hat{\beta}$ instead of β reflects the addition of a drift term in our representation of W under the change of measure associated with $\hat{M}$.

Assumption 6.1 *There exists a probability measure $\hat{Q}$ on $\mathbb{R}^n$ such that*

$$\int \hat{\mathcal{A}} f(x)\, d\hat{Q}(x) = 0$$

for all f that are twice continuously differentiable and have compact support.

By using $\hat{Q}$ to initialize the state, the process X is stationary under this change of measure. There are many well-known results for the existence of stationary distributions; see, for example [8], Chaps. 3 and 4. While there may be multiple solutions to the principal eigenvalue problem, [7] show that there is at most one solution for which the resulting probability measure makes X stationary and Harris-recurrent.

Associated with the Markov process X, there is a semigroup of conditional expectation operators, which may be extended to the space $\hat{L}^p$ of Borel-measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ with $\int |f(x)|^p\, d\hat{Q}(x) < \infty$.[6] Let $Z = \{f \in \hat{L}^2 : \int f(x)\, d\hat{Q}(x) = 0\}$.

[6] See, for example [6] for a discussion of the construction of the semigroup of conditional expectation operators in $\hat{L}^2$ and the construction of its associated generator.

Assumption 6.2 *The semigroup of conditional expectation operators associated with X under the change of measure implied by $\hat{M}$ and $\hat{Q}$ is a strong contraction semigroup on Z.*

As discussed by [11] and [6], under Assumption 6.2, the Markov process is ρ-mixing with mixing coefficients that necessarily decay exponentially to zero.

Proposition 6.3 *Suppose that $\gamma_d \cdot \kappa$, γ_d and $\frac{1}{e}$ are in $\hat{L}^2$. Then*

$$\lim_{t\to\infty} \rho_t(x) = -\int \gamma_d \cdot \kappa \, d\hat{Q}$$

in probability under the $\hat{Q}$-measure.

Thus, long-term risk prices are obtained by changing the state-dependent risk exposure γ_d in the representation given by Proposition 6.3. As in the local counterpart given in Proposition 4.2, we think of γ_d as parameterizing the exposure to (growth-rate) risk, which we allow to be state-dependent. The vector $(\kappa + \gamma_g)$ is the risk exposure of the martingale component of SG and γ_g is the risk exposure of the multiplicative martingale growth functional. In effect, the state-dependent vector κ in conjunction with the probability distribution $\hat{Q}$ determine the long-term counterpart to the local risk price vector $-\gamma_s$ given in Proposition 4.2.

Proof of Proposition 6.3 Recall that if $\hat{e} = \frac{1}{e}$, then

$$\rho_t(x) = \frac{\frac{1}{t}\hat{E}[\hat{e}(X_t)(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du + \int_0^t \gamma_d(X_u)\,d\hat{W}_u) \mid X_0 = x]}{\hat{E}[\hat{e}(X_t) \mid X_0 = x]}.$$

First notice that

$$\begin{aligned} &\frac{1}{t}\hat{E}\left[\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du + \int_0^t \gamma_d(X_u)\,d\hat{W}_u\right) \,\middle|\, X_0 = x\right] \\ &\quad = \frac{1}{t}\hat{E}\left[\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du \,\middle|\, X_0 = x\right] \longrightarrow \int \gamma_d\cdot\kappa\,d\hat{Q} \end{aligned} \tag{6.1}$$

in $\hat{L}^2$ under Assumption 6.2. $\hat{L}^2$-convergence implies convergence in $\hat{Q}$-probability.

Next, we show that

$$\begin{aligned} &\frac{1}{t}\hat{E}\Big[\big(\hat{e}(X_t) - \hat{E}\big[\hat{e}(X_t) \mid X_0 = x\big]\big) \\ &\quad\times\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du + \int_0^t \gamma_d(X_u)\,d\hat{W}_u\right) \,\Big|\, X_0 = x\Big] \longrightarrow 0 \end{aligned} \tag{6.2}$$

in $\hat{L}^1$. We consider this in two parts. First,

$$\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]\right)\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du\right) \,\middle|\, X_0 = x\right]$$
$$= \hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]\right)\right.$$
$$\left.\times \int_0^t \left(\gamma_d(X_u)\cdot\kappa(X_u) - \hat{E}\left(\gamma_d(X_u)\cdot\kappa(X_u)\right)\right) du \,\middle|\, X_0 = x\right].$$

Notice that

$$\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]\right)^2 \mid X_0 = x\right] \le \hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)\right]\right)^2 \mid X_0 = x\right]$$

and

$$\hat{E}\left[\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)\right]\right)^2 \mid X_0 = x\right]\right] \le \hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left(\hat{e}(X_t)\right)\right)^2\right] < \infty$$

since $\hat{e}(X_t)$ has finite second moment under the $\hat{Q}$-stationary distribution. The bound can be chosen to be independent of t. Moreover,

$$\hat{E}\left[\int_0^t \left(\gamma_d(X_u)\cdot\kappa(X_u) - \hat{E}\left[\gamma_d(X_u)\cdot\kappa(X_u)\right]\right) \,\middle|\, X_0 = x\right]$$

converges in $\hat{L}^2$ to a function of x with a finite second $\hat{Q}$-moment under Assumption 6.2. It follows from the Cauchy–Schwarz inequality that

$$\frac{1}{t}\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]\right)\left(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\,du\right) \,\middle|\, X_0 = x\right] \longrightarrow 0$$

in $\hat{L}^1$.

Second, consider

$$\frac{1}{t}\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]\right)\left(\int_0^t \gamma_d(X_u)\,d\hat{W}_u\right) \,\middle|\, X_0 = x\right]$$
$$\le \frac{1}{\sqrt{t}}\sqrt{\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)\right]\right)^2 \mid X_0 = x\right]}\sqrt{\hat{E}\left[\frac{1}{t}\int_0^t \left|\gamma_d(X_u)\right|^2 du \,\middle|\, X_0 = x\right]},$$

where the inequality is an application of the conditional Cauchy–Schwarz inequality. Provided that $\gamma_d(X_u)$ has a finite second moment under the $\hat{}$ distribution, the right-hand side converges to zero in $\hat{L}^1$ since the unconditional second moments of

$$\sqrt{\hat{E}\left[\left(\hat{e}(X_t) - \hat{E}\left[\hat{e}(X_t)\right]\right)^2 \mid X_0 = x\right)}$$

and

$$\sqrt{\hat{E}\left[\frac{1}{t}\int_0^t \left|\gamma_d(X_u)\right|^2 du \,\middle|\, X_0 = x\right]}$$

are finite and independent of t.

Given these two intermediate results, (6.2) follows. Finally,

$$\hat{E}\big[\hat{e}(X_t) \mid X_0 = x\big] \longrightarrow \int \hat{e}\, d\hat{Q} > 0$$

in $\hat{L}^2$. Thus,

$$\frac{1}{t} \frac{\hat{E}[(\hat{e}(X_t) - \hat{E}[\hat{e}(X_t)|X_0=x])(\int_0^t \gamma_d(X_u)\cdot\kappa(X_u)\, du + \int_0^t \gamma_d(X_u)\, d\hat{W}_u)|X_0=x]}{\hat{E}[\hat{e}(X_t)|X_0=x]}$$

tends to 0 in $\hat{Q}$-probability. The conclusion follows from this result combined with (6.1). □

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References

1. Bismut, J.-M.: Martingales, the Malliavin calculus and hypoellipticity under general Hormander's conditions. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 469–505 (1981)
2. Fournié, E., Lasry, J.-M., Lebuchoux, J., Lions, P.-L.: Applications of Malliavin calculus to Monte Carlo methods in finance. Finance Stoch. **3**, 391–412 (1999)
3. Fournié, E., Lasry, J.-M., Lebuchoux, J., Lions, P.-L.: Applications of Malliavin calculus to Monte Carlo methods in finance. II. Finance Stoch. **5**, 201–236 (2001)
4. Grigelionis, B., Mackevičius, V.: The finiteness of moments of a stochastic exponential. Stat. Probab. Lett. **64**, 243–248 (2003)
5. Hansen, L.P.: Modeling the long run: Valuation in dynamic stochastic economies. Fisher-Schultz Lecture at the European Meetings of the Econometric Society (2008). http://home.uchicago.edu/~lhansen/
6. Hansen, L.P., Scheinkman, J.A.: Back to the future: Generating moment implications for continuous-time Markov processes. Econometrica **63**, 767–804 (1995)
7. Hansen, L.P., Scheinkman, J.A.: Long-term risk: An operator approach. Econometrica **77**, 177–234 (2009)
8. Hasminskii, R.Z.: Stochastic Stability of Differential Equations, 2nd edn. Kluwer Academic, Dordrecht (1980)
9. Itô, K., Watanabe, T.: Transformation of Markov processes by multiplicative functionals. Ann. Inst. Fourier **15**, 13–30 (1965)
10. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, New York (1991)
11. Rosenblatt, M.: Markov Processes: Structure and Asymptotic Behavior. Springer, New York (1971)