

BACK TO THE FUTURE: GENERATING MOMENT IMPLICATIONS FOR CONTINUOUS-TIME MARKOV PROCESSES

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Continuous-time Markov processes can be characterized conveniently by their infinitesimal generators. For such processes there exist forward and reverse-time generators. We show how to use these generators to construct moment conditions implied by stationary Markov processes. Generalized method of moments estimators and tests can be constructed using these moment conditions. The resulting econometric methods are designed to be applied to discrete-time data obtained by sampling continuous-time Markov processes.

KEYWORDS: Continuous-time Markov processes, discrete-time sampling, diffusions, generalized method of moments, infinitesimal generator, nonparametric identification, reversibility.

1. INTRODUCTION

IN THIS PAPER WE DERIVE MOMENT CONDITIONS for estimating and testing continuous-time Markov models using discrete-time data. An extensive literature exists on estimating continuous time *linear* models from discrete-time data emanating from the work of A. W. Phillips (1959). This literature includes treatments of identification (e.g., see P. C. B. Phillips (1973), Hansen and Sargent (1983)) as well as estimation (e.g., see P. C. B. Phillips (1973), Robinson (1976), Harvey and Stock (1985)). Our aim is to develop new methods for estimation and inference that can be applied to continuous time *nonlinear* Markov models, again from the vantage point of discrete-time sampling. Recently there has been a considerable interest among economists in understanding the role of nonlinearities in dynamic models (see Scheinkman (1990) for a survey of this literature). Furthermore, several particular nonlinear continuous-time models have been proposed for the term structure of interest rates (e.g., see Cox, Ingersoll, and Ross (1985), Heath, Jarrow, and Morton (1992)) and for exchange rates (e.g., see Froot and Obstfeld (1991) and Krugman (1991)). Among other things, we develop tools for assessing the empirical plausibility of these models.

Likelihood-based methods of estimation and inference for nonlinear continuous time models can be very difficult to implement due to the computational costs associated with evaluating the likelihood function (e.g., see Lo (1988)). This is true even when the Markov state vector is completely observable at any

¹ Conversations with Buz Brock, Henri Berestycki, Darrell Duffie, Ivar Ekeland, Hedi Kallal, Carlos Kening, Jean Michel Lasry, Pierre Louis Lions, Andy Lo, Erzo Luttmer, Ellen McGratten, Jesus Santos, Nizar Touzi, and Arnold Zellner are gratefully acknowledged. We received helpful comments from an editor and three anonymous referees on an earlier version of this paper. Thanks to Andy Lo and Steven Spielberg for suggesting the title. Portions of this research were funded by grants from the National Science Foundation.

point in time, as we assume here. The reason for this difficulty is that a continuous-time Markov process is typically specified in terms of its local evolution. Evaluating the discrete-time transition density can be quite costly because it may require solving numerically a partial differential equation as in the case of a diffusion. The computational costs can become excessive because these calculations must be repeated for each hypothetical parameter value and each observed state.²

In this paper we adopt a more pragmatic approach. We begin by considering a Markov process specified in terms of its *infinitesimal generator*. Formally, this generator is defined as an operator on a function space, and, in effect, this operator stipulates the local evolution of the process. For instance, for a diffusion model specified as a solution to a stochastic differential equation, the generator can be constructed from the coefficients of the differential equation and the associated boundary conditions.

Given the infinitesimal generator, we show how to construct two sets of moment conditions that are often easy to compute in practice. As a consequence, both sets of moment conditions can be used to construct generalized-method-of-moments estimators of an unknown parameter vector and diagnostic tests. One set involves only the contemporaneous Markov state vector and hence uses only the marginal distribution of that vector. The second set includes functions of the state vector in two adjacent time periods, and hence, like the score vector from a likelihood function, exploits properties of the conditional distribution of the current state vector given the past.

Moment conditions in the second set are most conveniently represented in terms of the original generator as well as the *reverse-time generator* for a process running *backwards* in time. Although there exist general characterizations of reverse-time generators, the second set of moment conditions may be easiest to apply when the underlying Markov processes under consideration are (time) reversible. As a consequence, we use results in the probability theory literature to show that a potentially rich collection of models are reversible, including many multi-factor models of the term structure of interest rates that have been suggested in the literature.

Another strategy that has been proposed for estimating continuous time Markov processes is to use numerical methods to approximate moments. Duffie and Singleton (1993) and Gourieroux, Monfort, and Renault (1992) suggested the use of simulation while He (1990) proposed the use of binomial approximations. An attractive feature of the simulation approach is that the Markov state vector does not have to be fully observed. However, for both of these approaches it may be difficult to account for the magnitude of the approximation

² Exceptions to this are the nonlinear Markov process assumed by Cox, Ingersoll, and Ross (1985) in their analysis of the term structure of interest rates and the reflecting barrier model of exchange rates assumed by Krugman (1991). The transition densities for this process have been fully characterized (e.g., see Feller (1951), Wong (1964), and Levy (1993)). However, small departures in the form of the nonlinearities make likelihood-based methods much more numerically intensive to implement.

error, and for some applications it may be numerically costly to ensure that the approximation error is small.

When alternative parametric estimation methods are computationally feasible, the moment conditions we derive can be used to construct diagnostic tests of the model which are easy to implement in practice. Moreover, they permit the estimated model to be nested within a larger class of alternative continuous time models including ones whose transition probabilities may be much more difficult to compute in practice. Our interest in this approach extends beyond consideration of computational costs vis-a-vis other methods of estimating parametric models, however. For instance, we also study nonparametric identification via these moment conditions. It is well known that identifying continuous-time models from discrete-time data can be problematic because of what is known as the *aliasing phenomenon*: distinct continuous-time processes may look identical when sampled at regular time intervals. Using spectral representation theory for self-adjoint operators, we show that there is no aliasing problem when it is known that the continuous-time process is reversible. Although the generalized-method-of-moments approach described in our paper is designed to be computationally tractable, complete (nonparametric) identification of reversible processes is not possible using the moment conditions we derive. However, we do show that in the context of scalar diffusion models, the local mean (drift) and local variance (diffusion coefficient) can be identified up to a common scale factor.

The focus of this paper is on deriving moment conditions implied by infinitesimal generators and on characterizing the extent to which these moment conditions can discriminate among the members of a class of infinitesimal generators. In addition, we provide restrictions on infinitesimal generators that ensure that the Law of Large Numbers and Central Limit Theorem apply to a discrete-sampled process. Armed with these approximation results, we can apply the results in Hansen (1982) to justify formally estimation and inference using generalized method of moments. On the other hand, we do not address formally issues of statistical efficiency and nonparametric estimation and inference using the moment conditions we derive. Such issues are deferred to future work.

This paper is organized as follows. In Section 2 we review the mathematical construction of an infinitesimal generator for a continuous time Markov process and describe properties of the generator that are important for our analysis. Since leading examples of continuous time Markov processes are scalar diffusion, we use these processes to illustrate many of the results in this paper. Consequently, in this section we recall the well known connection between the generator of a scalar diffusion and its local mean and variance. In Section 3 we show how to use the infinitesimal generator to construct two families of moment conditions expressed in terms of the Markov state vector. Other examples of Markov processes together with their infinitesimal generators are presented in Section 4. We study the observable implications of each family of moment conditions in Section 5. For instance, we show that our first family of moment

conditions can be used to distinguish alternative candidate generators that imply distinct marginal distributions for the Markov state vector. We also provide a characterization of the additional informational content provided by the second family of moment conditions obtained by reversing calendar time. Since the domains of the generators are sometimes difficult to characterize fully, in Section 6 we show how to reduce the family of test functions used in the moment conditions to include only ones for which the generator can be represented in a convenient manner. For the moment conditions to be of use in practice, we must be able to approximate expectations of functions of the state vector using a discrete-time moment analog. In Section 7 we present sufficient conditions for these approximations to be valid. To facilitate verification, these conditions are expressed as restrictions on the infinitesimal generator. These large-sample approximations can also be used to justify other estimation methods than the one described in this paper.

2. INFINITESIMAL GENERATORS

In this section we give the mathematical basis for our analysis. The focal point is on the construction of the *infinitesimal generator* of a strictly stationary, continuous-time, n -dimensional, vector Markov process $\{x_t\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P}r)$.

Let \mathcal{Q} be the probability measure induced on \mathbf{R}^n by x_t (for any t), $\mathcal{L}^2(\mathcal{Q})$ be the space of all Borel measurable functions $\phi: \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\int_{\mathbf{R}^n} \phi^2 d\mathcal{Q} < \infty,$$

and let $\langle \cdot | \cdot \rangle$ and $\|\cdot\|$ be the usual inner product and norm on $\mathcal{L}^2(\mathcal{Q})$. Associated with the Markov process is a family of operators $\{\mathcal{T}_t: t \geq 0\}$ where for each $t \geq 0$, \mathcal{T}_t is defined by³

$$(2.1) \quad \mathcal{T}_t \phi(y) \equiv E[\phi(x_t) | x_0 = y].$$

This family is known to satisfy several properties. For instance, from Nelson (1958, Theorem 3.1) it follows that for each $t \geq 0$, the following properties hold:

PROPERTY P1: $\mathcal{T}_t: \mathcal{L}^2(\mathcal{Q}) \rightarrow \mathcal{L}^2(\mathcal{Q})$ is well defined, i.e., if $\phi = \psi$ with \mathcal{Q} probability one, then $\mathcal{T}_t \phi = \mathcal{T}_t \psi$ with \mathcal{Q} probability one and for each $\phi \in \mathcal{L}^2(\mathcal{Q})$, $\mathcal{T}_t \phi \in \mathcal{L}^2(\mathcal{Q})$.

PROPERTY P2: $\|\mathcal{T}_t \phi\| \leq \|\phi\|$ for all $\phi \in \mathcal{L}^2(\mathcal{Q})$, i.e., \mathcal{T}_t is a (weak) contraction.

PROPERTY P3: For any $s, t \geq 0$, $\mathcal{T}_{t+s} = \mathcal{T}_t \mathcal{T}_s$, i.e., $\{\mathcal{T}_t: t \geq 0\}$ is a (one-parameter) semigroup.

³ Throughout this paper we follow the usual convention of not distinguishing between an equivalence class and the functions in the equivalence class. Moreover, in (2.1) we are abusing notation in a familiar way, e.g. see Chung (1974, pp. 299 and 230).

Property P2 is the familiar result that the conditional expectation operator can only reduce the second moment of a random variable. Property P3 is implied by the Law of Iterated Expectations since the expectation of $\phi(x_{t+s})$ given x_0 can be computed by first conditioning on information available at time s .

Our approach to exploring the implications of continuous-time Markov models for discrete time data is to study limits of expectations over small increments of time. In order for this approach to work, we impose a *mild* restriction on the smoothness properties of $\{\mathcal{T}_t\}$.

ASSUMPTION A1: For each $\phi \in \mathcal{L}^2(\mathcal{Q})$, $\{\mathcal{T}_t\phi : t \geq 0\}$ converges [in $\mathcal{L}^2(\mathcal{Q})$] to ϕ as $t \downarrow 0$.

Assumption A1 is implied by measurability properties of the underlying stochastic process $\{x_t\}$. Recall that $\{x_t\}$ can always be viewed as a mapping from $\mathbf{R} \times \Omega$ into \mathbf{R}^n . Form the product sigma algebra using the Borelians of \mathbf{R} and the events of Ω . A sufficient condition for Assumption A1 is that the mapping defined by the stochastic process be Borel measurable with respect to the product sigma algebra.⁴

For some choices of $\phi \in \mathcal{L}^2(\mathcal{Q})$, the family of operators is *differentiable* at zero, i.e. $\{[\mathcal{T}_t\phi - \phi]/t : t > 0\}$ has an $\mathcal{L}^2(\mathcal{Q})$ limit as t goes to 0. Whenever this limit exists, we denote the limit point $\mathcal{A}\phi$. We refer to \mathcal{A} as the *infinitesimal generator*. The domain \mathcal{D} of this generator is the family of functions ϕ in $\mathcal{L}^2(\mathcal{Q})$ for which $\mathcal{A}\phi$ is well defined. Typically, \mathcal{D} is a proper subset of $\mathcal{L}^2(\mathcal{Q})$. Since \mathcal{A} is the *derivative* of $\{\mathcal{T}_t : t \geq 0\}$ at $t = 0$, and $\{\mathcal{T}_t : t \geq 0\}$ is a semigroup, \mathcal{A} and \mathcal{T}_t commute on \mathcal{D} . Moreover the following properties are satisfied (e.g., see Pazy (1983, Theorem 2.4, p. 4)):

PROPERTY P4: For any $\phi \in \mathcal{L}^2(\mathcal{Q})$, $\int_0^t \mathcal{T}_s \phi \, ds \in \mathcal{D}$ and $\mathcal{A} \int_0^t \mathcal{T}_s \phi \, ds = \mathcal{T}_t \phi - \phi$.

PROPERTY P5: For any $\phi \in \mathcal{D}$, $\mathcal{T}_t \phi - \phi = \int_0^t \mathcal{A}[\mathcal{T}_s \phi] \, ds = \int_0^t \mathcal{T}_s[\mathcal{A}\phi] \, ds$.

Property P4 gives the operator counterpart to the familiar relation between derivatives and integrals. Property P5 extends the formula by interchanging the order of integration and applications of the operators \mathcal{T}_s and \mathcal{A} .

There are three additional well known properties of infinitesimal generators of continuous-time Markov operators that we will use in our analysis:

PROPERTY P6: \mathcal{D} is dense in $\mathcal{L}^2(\mathcal{Q})$.

⁴ It follows from Halmos (1974, p. 148) that $\mathcal{T}_t\phi : t \geq 0$ is weakly measurable for any $\phi \in \mathcal{L}^2(\mathcal{Q})$ and hence from Theorem 1.5 of Dynkin (1965, p. 35) that A1 is satisfied.

PROPERTY P7: \mathcal{A} is a closed linear operator, i.e. if $\{\phi_n\}$ in \mathcal{D} converges to ϕ_0 and $\{\mathcal{A}\phi_n\}$ converges to ψ_0 , then ϕ_0 is in \mathcal{D} and $\mathcal{A}\phi_0 = \psi_0$.

PROPERTY P8: For every $\lambda > 0$, $\lambda I - \mathcal{A}$ is onto.

(A reference for P6 and P7 and P8 is Pazy (1983, Chapter 1, Corollary 2.5 and Theorem 4.3).)

A final property of \mathcal{A} that will be of value to us is P9:

PROPERTY P9: $\langle \phi | \mathcal{A}\phi \rangle \leq 0$ for all $\phi \in \mathcal{D}$, i.e. \mathcal{A} is quasi negative semidefinite.⁵

This property follows from the (weak) contraction property of $\{\mathcal{T}_t\}$ and the Cauchy-Schwarz Inequality since for any $t > 0$,

$$(2.2) \quad \langle \phi | \mathcal{T}_t \phi - \phi \rangle \leq \|\phi\|(\|\mathcal{T}_t \phi\| - \|\phi\|) \leq 0.$$

In modeling Markov processes one may start with a candidate infinitesimal generator satisfying a particular set of properties, and then show that there exists an associated Markov process (e.g., see Corollary 2.8 of Ethier and Kurtz (1986, p. 170)). Furthermore, combinations of the above properties are sufficient for \mathcal{A} to be the infinitesimal generator of a semi-group of contractions satisfying A1. For instance, it suffices that P6 and P9 holds as well as P8 for some $\lambda > 0$. This is part of the Lumer-Phillips Theorem (e.g., see Pazy (1983, Chapter 1, Theorem 4.3)).

In many economic and finance applications it is common to start with a stochastic differential equation, instead of specifying the infinitesimal generator directly. As the example below shows, there are well known connections between the coefficients of the stochastic differential equation and the infinitesimal generator.

EXAMPLE 1 (Scalar Diffusion Process): Suppose that $\{x_t\}$ satisfies the stochastic differential equation:

$$(2.3) \quad dx_t = \mu(x_t) dt + \sigma(x_t) dW_t$$

where $\{W_t : t \geq 0\}$ is a scalar Brownian motion. As we elaborate in the discussion below, under some additional regularity conditions the generator for this process can be represented as

$$\mathcal{A}\phi = \mu\phi' + (1/2)\sigma^2\phi''$$

on a large subset of its domain.

Prior to justifying this representation, we consider sufficient conditions for the existence of a unique stationary solution to the stochastic differential equation. There are many results in the literature that establish the existence and uniqueness of a Markov process $\{x_t\}$ satisfying (2.3) for a prespecified

⁵ When a generator of a semigroup defined on an arbitrary Banach space satisfies the appropriate generalization of P9, it is referred to as being *dissipative* (e.g., see Pazy (1983, p. 13)).

initialization. One set of sufficient conditions, that will be exploited in Section 6, requires that the *diffusion coefficient* σ^2 be strictly positive with a bounded and continuous second derivative and that the *local mean* μ have a bounded and continuous first derivative. These conditions also imply that $\{x_t\}$ is a *Feller* process, that is, for any continuous function ϕ , $\mathcal{F}\phi$ is continuous.⁶

Of course, for the process to be stationary, we must initialize appropriately. An additional restriction is required for there to exist such an initialization. We follow Karlin and Taylor (1981) and others by introducing a *scale function* S and its derivative:

$$s(y) \equiv \exp\left\{-\int^y 2[\mu(z)/\sigma^2(z)] dz\right\};$$

and a *speed density* $1/s\sigma^2$, which we assume to be integrable.⁷ If we use the measure \mathcal{Q} with density

$$(2.4) \quad q(y) = \frac{1}{s(y)\sigma^2(y) \int [s(z)\sigma^2(z)]^{-1} dz}$$

to initialize the process, then the Markov process $\{x_t\}$ generated via the stochastic differential equation (2.3) will be stationary. In fact, under these hypotheses \mathcal{Q} is the unique stationary distribution that can be associated with a solution of (2.3).

The infinitesimal generator is defined on a subspace of $\mathcal{L}^2(\mathcal{Q})$ that contains at least the subset of functions ϕ for which ϕ' and ϕ'' are continuous, $\sigma\phi' \in \mathcal{L}^2(\mathcal{Q})$, and such that the second-order differential operator:

$$(2.5) \quad L\phi(y) \equiv \mu(y)\phi'(y) + (1/2)\sigma^2(y)\phi''(y)$$

yields an $\mathcal{L}^2(\mathcal{Q})$ function. Ito's Lemma implies that $\{\phi(x_t) - \phi(x_0) - \int_0^t L\phi(x_s) ds\}$ is a continuous martingale. Take expectations conditional on x_0 , to obtain

$$(2.6) \quad E[\phi(x_t)|x_0] - \phi(x_0) = E\left[\int_0^t L\phi(x_s) ds|x_0\right].$$

Or by Fubini's Theorem,

$$[\mathcal{F}_t\phi(x_0) - \phi(x_0)]/t = (1/t)\int_0^t \mathcal{F}_s L\phi(x_0) ds.$$

Using the continuity property of \mathcal{F}_s and applying the Triangle Inequality we conclude that $\mathcal{A}\phi = L\phi$.

⁶ For alternative Lipschitz and growth conditions see the hypotheses in Theorem 3.2, p. 79 or the weaker local hypotheses of Theorem 4.1, p. 84 of Has'minskii (1980). Alternatively, the Yamada-Watanabe Theorem in Karatzas and Shreve (1988, p. 291) can be applied.

⁷ It suffices for integrability that for some strictly positive C and K and $|y| \geq K$, $\mu y + \sigma^2/2 \leq -C$. This can be verified by using this inequality to construct an integrable upper bound for the speed density or by appealing to more general results in Has'minsky (1980).

3. MOMENT CONDITIONS

In this section we characterize two sets of moment conditions that will be of central interest for our analysis. The basic idea underlying both sets of moment conditions is to exploit the fact that expectations of time invariant functions of say x_{t+1} and x_t do not depend on calendar time t . We connect this invariance to the local specification of the underlying Markov process, or more precisely to the infinitesimal generator. Formally, these moment conditions are derived from two important (and well known) relations. The first relation links the stationary distribution \mathcal{Q} and the infinitesimal generator \mathcal{A} , and the second one exploits the fact that \mathcal{A} and \mathcal{T}_1 commute. Much of our analysis will focus on a fixed sampling interval which we take to be one. From now on we write \mathcal{T} instead of \mathcal{T}_1 .

Since the process $\{x_t\}$ is stationary, $E[\phi(x_t)]$ is independent of t , implying that its derivative with respect to t is zero. To see how this logic can be translated into a set of moment conditions, note that by the Law of Iterated Expectations, the expectation of $\phi(x_t)$ can either be computed directly or by first conditioning on x_0 :

$$(3.1) \quad \int_{\mathbb{R}^n} \phi d\mathcal{Q} = \int_{\mathbb{R}^n} \mathcal{T}_1 \phi d\mathcal{Q} \quad \text{for all } \phi \in \mathcal{L}^2(\mathcal{Q}).$$

Hence for any $\phi \in \mathcal{D}$, we have that

$$(3.2) \quad \int_{\mathbb{R}^n} \mathcal{A}\phi d\mathcal{Q} = \lim_{t \downarrow 0} (1/t) \int_{\mathbb{R}^n} [\mathcal{T}_t \phi - \phi] d\mathcal{Q} = 0$$

since (3.1) holds for all positive t and $\{(1/t)[\mathcal{T}_t \phi - \phi] : t > 0\}$ converges in $\mathcal{L}^2(\mathcal{Q})$ to $\mathcal{A}\phi$. Relation (3.2) shows the well known link between the generator \mathcal{A} and the stationary distribution \mathcal{Q} (e.g., see Ethier and Kurtz (1986, Proposition 9.2, p. 239)). Hence our first set of moment conditions is as follows.

$$\text{C1: } E[\mathcal{A}\phi(x_t)] = 0 \text{ for all } \phi \in \mathcal{D}.$$

The stationarity of $\{x_t\}$ also implies that $E[\phi(x_{t+1})\phi^*(x_t)]$ does not depend on calendar time. Rather than deriving moment conditions by exploiting this invariance directly, a more convenient derivation is to start by using the fact that \mathcal{A} and \mathcal{T} commute. That is,

$$(3.3) \quad E[\mathcal{A}\phi(x_{t+1})|x_t = y] = \mathcal{A}\{E[\phi(x_{t+1})|x_t = y]\} \text{ for all } \phi \in \mathcal{D}.$$

It may be difficult to evaluate the right side of (3.3) in practice because it entails computing the conditional expectation of $\phi(x_{t+1})$ prior to the application of \mathcal{A} . For this reason, we also derive an equivalent set of unconditional moment restrictions which are often easier to use. These moment restrictions are representable using the semigroup and generator associated with the reverse-time Markov process.

The semigroup $\{\mathcal{T}_t^*\}$ associated with the reverse-time process is defined via

$$(3.4) \quad \mathcal{T}_t^* \phi^*(y) \equiv E[\phi^*(x_0) | x_t = y].$$

The family $\{\mathcal{T}_t^* : t \geq 0\}$ is also a contraction semigroup of operators satisfying continuity restriction A1. Let \mathcal{A}^* denote the infinitesimal generator for this semigroup with domain \mathcal{D}^* . The operator \mathcal{T}_t^* is the adjoint of \mathcal{T}_t and \mathcal{A}^* is the operator adjoint of \mathcal{A} . To verify these results, note that it follows from (2.1), (3.4), and the Law of Iterated Expectations that

$$(3.5) \quad \begin{aligned} \langle \phi^* | \mathcal{T}_t \phi \rangle &= E\{\phi^*(x_0) E[\phi(x_t) | x_0]\} \\ &= E[\phi^*(x_0) \phi(x_t)] \\ &= E\{\phi(x_t) E[\phi^*(x_0) | x_t]\} \\ &= \langle \phi | \mathcal{T}_t^* \phi^* \rangle. \end{aligned}$$

We can now use Corollary 10.6 of Pazy (1983, p. 41) to show that \mathcal{A}^* is the operator adjoint of \mathcal{A} and vice versa. A process is reversible if the generator \mathcal{A} is self adjoint, i.e. $\mathcal{A} = \mathcal{A}^*$.

For any ϕ in the domain \mathcal{D} of \mathcal{A} and ϕ^* in $\mathcal{L}^2(\mathcal{C})$, the fact that \mathcal{A} and \mathcal{T} commute implies that

$$(3.6) \quad \langle \mathcal{T} \mathcal{A} \phi | \phi^* \rangle - \langle \mathcal{A} \mathcal{T} \phi | \phi^* \rangle = 0.$$

When ϕ^* is restricted to be in the domain \mathcal{D}^* of \mathcal{A}^* ,

$$(3.7) \quad \langle \mathcal{A} \mathcal{T} \phi | \phi^* \rangle = \langle \phi | \mathcal{T}^* \mathcal{A}^* \phi^* \rangle.$$

Substituting (3.6) into (3.7) and applying the Law of Iterated Expectations we find the following.

$$C2: E[\mathcal{A} \phi(x_{t+1}) \phi^*(x_t) - \phi(x_{t+1}) \mathcal{A}^* \phi^*(x_t)] = 0 \text{ for all } \phi \in \mathcal{D} \text{ and } \phi^* \in \mathcal{D}^*.$$

Since \mathcal{A}^* enters C2, these moment conditions exploit both the forward- and reverse-time characterization of the Markov process. As we hinted above, these moment conditions can be interpreted as resulting from equating the time derivative of $E[\phi(x_{t+1}) \phi^*(x_t)]$ to zero.

When ϕ^* is a constant function, $\mathcal{A}^* \phi^*$ is identically zero and moment condition C2 collapses to C1. In our subsequent analysis, we will find it convenient to look first at moment conditions in the class C1, and then to ascertain the incremental contribution of the class C2. Moreover, for Markov processes that are not reversible ($\mathcal{A} \neq \mathcal{A}^*$), the additional moment conditions in the class C2 can be more difficult to use in practice because they use the reverse-time generator. The more extensive set of moment conditions are still of interest, however, because sometimes it can be shown that the processes of interest are reversible. Even when they are not, it is typically easier to compute the reverse-time generator than the one-period transition distribution.

The stationary scalar diffusions described previously in Example 1 are reversible. This is easy to verify by following the usual approach of introducing the integrating factor $1/q$, and writing

$$(3.8) \quad \mathcal{A}\phi = (1/q)(\sigma^2 q \phi' / 2)'.$$

Let \mathcal{E}_K^2 denote the space of functions that are twice continuously differentiable with a compact support. Integration by parts implies that for functions ψ and ϕ in \mathcal{E}_K^2 ,

$$(3.9) \quad \langle \mathcal{A}\phi | \psi \rangle = \langle \phi | \mathcal{A}\psi \rangle.$$

As we will discuss in Section 6, if (3.9) holds for ψ and ϕ in \mathcal{E}_K^2 , it actually holds for each ψ and ϕ in \mathcal{D} ; that is, \mathcal{A} is symmetric. Since \mathcal{A} is the generator of a contraction semigroup in a Hilbert space, it is self adjoint and $\{x_t\}$ is reversible (see Proposition VII.6 on p. 113 of Brezis (1983)).

One strategy for using these moment conditions for estimation and inference is as follows. Suppose the problem confronting an econometrician is to determine which, if any, among a parameterized set of infinitesimal generators is compatible with a discrete-time sample of the process $\{x_t\}$. For instance, imagine that the aim is to estimate the "true parameter value," say β_o , associated with a parameterized family of generators \mathcal{A}_β for β in some admissible parameter space. Vehicles for accomplishing this task are the sample counterparts to moment conditions C1 and C2. By selecting a finite number of test functions ϕ , the unknown parameter vector β_o can be estimated using generalized method of moments (e.g., see Hansen (1982)) and the remaining over-identifying moment conditions can be tested. In Section 5 of this paper we characterize the information content of each of these two sets of moment conditions for discriminating among alternative sets of infinitesimal generators; and in Section 7 we supply some supporting analysis for generalized method of moments estimation and inference by deriving some sufficient conditions on the infinitesimal generators for the Law of Large Numbers and Central Limit Theorem to apply.

For particular families of scalar diffusions and test functions, moment conditions in the class C1 have been used previously, albeit in other guises. For instance Wong (1964) showed that first-order polynomial specifications of μ , and second-order polynomial specifications of σ^2 are sufficient to generate processes with stationary densities in the Pearson class. Pearson's method of moment estimation of these densities can be interpreted, except for its assumption that the data generation is i.i.d., as appropriately parameterizing the polynomials defining the drift and diffusion coefficients and using polynomial test functions in C1. This approach has been extended to a broader class of densities by Cobb, Koopstein, and Chen (1983). These authors suggested estimating higher-order polynomial specifications of μ for a pre-specified (second-order) polynomial specification of σ^2 . Their estimation can again be interpreted as using moment conditions C1 with polynomial test functions. Of course moment conditions C1 and C2 are easy to apply for other test functions

ϕ and ϕ^* whose first and second derivatives can be computed explicitly. Although Cobb, Koopstein, and Chen (1983) prespecify the local variance to facilitate identification, this assumption is not convenient for many applications in economics and finance. As we will see in Section 5, moment conditions in the class C2 can be used to help identify and estimate unknown parameters of the local variance.

4. EXAMPLES

In this section we give additional illustrations of infinitesimal generators for continuous-time Markov processes.

EXAMPLE 2 (Scalar Diffusions with Boundary Conditions): Many economic examples deal with processes that are restricted to a finite interval or to the nonnegative reals. The reasoning in Example 1 applies immediately to processes defined on an interval when both end points are entrance boundaries. Moreover, this analysis also can be extended to processes with reflecting boundaries as the ones assumed in the literature on exchange rate bands. Typically, a process with reflecting boundaries in an interval (ℓ, u) is constructed by changing equation (2.3) to include an appropriate additional term that is “activated” at the boundary points:

$$dx_t = \mu(x_t) dt + \sigma(x_t) dW_t + \theta(x_t) d\kappa_t$$

where $\{\kappa_t\}$ is a particular nondecreasing process that increases only when x_t is at the boundary, and

$$\theta(\ell) = 1 \quad \text{and} \quad \theta(u) = -1.$$

In addition we assume that the original μ and σ^2 define a *regular* diffusion, that is the functions s and $1/\sigma^2 s$ are integrable. We will also assume that s is bounded away from zero.⁸ Ito's Lemma applies to such processes with an additional term $\phi'(x_t)\theta(x_t)d\kappa_t$. This term vanishes if $\phi'(\ell) = \phi'(u) = 0$, and $\mathcal{A}\phi$ is again given as (2.5) or (3.8) for ϕ 's that satisfy these additional restrictions. To check that \mathcal{A} is self adjoint, it is now sufficient to verify (3.9) for ψ and ϕ that are twice continuously differentiable with first derivatives that vanish at the boundaries. When the drift coefficient has continuous first derivatives and diffusion coefficient continuous second derivatives, the right-hand side of (2.4) again defines the unique stationary distribution associated with the reflexive barrier process. The case with one reflecting barrier and the other one nonattracting can be handled in an analogous fashion.

EXAMPLE 3 (Multivariate Factor Models): We have just seen that many scalar diffusions are reversible. Reversibility carries over to multivariate diffusions built up as time invariant functions of a vector of independent scalar diffusions.

⁸ If σ^2 is bounded away from zero, all these assumptions hold but they also hold for some processes where σ^2 vanishes at the boundary.

More precisely, let $\{f_t\}$ be a vector diffusion with component processes that are independent, stationary, and reversible. Think of the components of $\{f_t\}$ as independent unobservable “factors.” Suppose that the observed process is a time invariant function of the factors:

$$x_t = F(f_t)$$

for some function F . Factor models of bond prices like those proposed by Cox, Ingersoll, and Ross (1985), Longstaff and Schwartz (1992), and Frachot, Janci, and Lacoste (1992) all have this representation. The resulting $\{x_t\}$ process will clearly be stationary as long as the factors are stationary. Moreover, since $\{f_t\}$ is reversible, so is $\{x_t\}$. To ensure that $\{x_t\}$ is a Markov process, we require that the factors can be recovered from the observed process. That is, F must be one-to-one. Finally, to guarantee that $\{x_t\}$ satisfies a stochastic differential equation, we restrict F to have continuous second derivatives. For a more general characterization of multivariable reversible diffusions, see Kent (1978).

EXAMPLE 4 (Multivariate Diffusion Models): More generally, suppose that $\{x_t\}$ satisfies the stochastic differential equation

$$dx_t = \mu(x_t) dt + \Sigma(x_t)^{1/2} dW_t$$

where $\{W_t : t \geq 0\}$ is an n -dimensional Brownian motion. Entry i of the local mean μ , denoted μ_i , and entry (i, j) of the diffusion matrix Σ , denoted σ_{ij} , are functions from R^n into R . The functions μ_i and the entries of $\Sigma(x_t)^{1/2}$ are assumed to satisfy Lipschitz conditions. (See Theorem 3.2, p. 79; the weaker local hypothesis of Theorem 4.1, page 84 of Has’minsky (1980); or Theorem 3.7, p. 297, Theorem 3.11, p. 300, and Remark 2.7 on p. 374 of Ethier and Kurz (1986).) The existence of a unique stationary distribution is assured if there exists a K such that $\Sigma(y)$ is positive definite for $|y| \leq K$ and for $|y| \geq K$,

$$\mu(y) \cdot Vy + (1/2) \text{trace} [\Sigma(y)V] < -1$$

for some positive definite matrix V . Moreover this guarantees that the process is mean recurrent. (See Corollary 1, p. 99; Example 1, p. 103; and Corollary 2, p. 123 of Has’minsky (1980)). If $\Sigma(y)$ is positive definite for all $y \in R^n$, then the density q is given by the unique nonnegative bounded solution q to the partial differential equation:

$$\begin{aligned} & - \sum_{j=1}^n \partial/\partial y_j [\mu_j(y)q(y)] + (1/2) \sum_{i=1}^n \sum_{j=1}^n (\partial^2/\partial y_i \partial y_j) [\sigma_{ij}(y)q(y)] \\ & = 0 \end{aligned}$$

that integrates to one. (See Has’minsky (1980, Lemma 9.4, p. 138).) Again by initializing the process using the measure \mathcal{Q} with density q , we construct a stationary Markov process $\{x_t\}$.

The infinitesimal generator is defined on a subspace of $\mathcal{L}^2(\mathcal{Q})$ that contains at least the space of functions ϕ for which $\partial\phi/\partial y$ and $\partial^2\phi/\partial y \partial y'$ are

continuous and the entries of $\Sigma^{1/2}\partial\phi/\partial y \in \mathcal{L}^2(\mathcal{E})$ via

$$(4.1) \quad \mathcal{A}\phi(y) = \mu(y) \cdot \partial\phi(y)/\partial y + (1/2) \text{trace} \left[\Sigma(y) \partial^2\phi(y)/\partial y \partial y' \right],$$

if the right-hand side of (4.1) is in $\mathcal{L}^2(\mathcal{E})$. Under suitable regularity conditions, a time-reversed diffusion is still a diffusion and the adjoint \mathcal{A}^* can be represented as⁹

$$(4.2) \quad \mathcal{A}^*\phi^*(y) = \mu^*(y) \cdot \partial\phi^*(y)/\partial y + (1/2) \text{trace} \left[\Sigma^*(y) \partial^2\phi^*(y)/\partial y \partial y' \right].$$

In (4.2) the diffusion matrix Σ^* turns out to be equal to Σ ; however, the local mean μ^* may be distinct from μ . Let Σ_j denote column j of Σ . It follows from Nelson (1958), Anderson (1982), Hausmann and Pardoux (1985), or Millet, Nualart, and Sanz (1989) that

$$\mu^*(y) = -\mu(y) + [1/q(y)] \sum_j \partial/\partial y_j [q(y) \Sigma_j(y)].$$

Therefore, on a dense subset of its domain the adjoint \mathcal{A}^* can be constructed from knowledge of μ , Σ , and the stationary density q .

Moment conditions in the class C1 remain easy to apply to test functions whose two derivatives can be readily computed. For multivariate diffusions that are not time reversible, it is, in general, much more difficult to calculate the reverse-time generator used in C2. An alternative approach is to approximate the reverse-time generator using a nonparametric estimator of the logarithmic derivative of the density. With this approach, nonparametric estimators of the density and its derivative appear in the constructed moment conditions even though the underlying estimation problem is fully parametric. The nonparametric estimator is used only as a device for simplifying calculations. Rosenblatt (1969) and Roussas (1969) described and justified nonparametric estimators of densities for stationary Markov processes. Moreover, estimation using moment conditions constructed with nonparametric estimates of functions such as $\partial[\log q(y)]/\partial y$ has been studied in the econometrics literature (e.g., see Gallant and Nychka (1987), Powell, Stock, and Stoker (1989), Robinson (1989), Chamberlain (1992), Newey (1993), Lewbel (1994)). Presumably results from these literatures could be extended to apply to our estimation problem.

EXAMPLE 5 (Markov Jump Process): Let η be a nonnegative bounded function mapping R^n into R and let $\pi(y, \Gamma)$ denote a transition function in the Cartesian product of R^n and the Borelians of R^n . Imagine the following stochastic process $\{x_t\}$. Dates at which changes in states occur are determined by a Poisson process with parameter $\eta(y)$ if the current state is y . Given that a change occurs, the transition probabilities are given by $\pi(y, \cdot)$. Hence we can think of a jump process as a specification of an underlying discrete time Markov chain with transition probabilities.

⁹ For example, Millet, Nualart, and Sanz (1989) showed that it suffices for coefficients μ and Σ of the diffusion to be twice continuously differentiable, to satisfy Lipschitz conditions, and for the matrix Σ to be uniformly nonsingular (see Theorem 2.3 and Proposition 4.2).

Additional restrictions must be imposed for this process to be stationary. First, suppose there exists a nonzero Borel measure $\tilde{\mathcal{Q}}$ satisfying the equation

$$\tilde{\mathcal{Q}}(\Gamma) = \int \pi(y, \Gamma) d\tilde{\mathcal{Q}}(y) \quad \text{for any Borelian } \Gamma.$$

In other words $\tilde{\mathcal{Q}}$ is the stationary distribution of the underlying discrete time Markov chain. If the Poisson intensity parameter η is constant, then the Markov jump process also has $\tilde{\mathcal{Q}}$ as a stationary distribution because the randomization of the jump times is independent of the current Markov state. In the more general case in which η depends on the state, there is an observationally equivalent representation with a constant Poisson parameter and a different underlying Markov chain (e.g., see Ethier and Kurtz (1986, p. 163)). The alternative Poisson intensity parameter is $\hat{\eta} \equiv \sup \eta$ and the transition probabilities for the alternative Markov chain are

$$\hat{\pi}(y, \Gamma) \equiv (1 - \eta/\hat{\eta}) \delta_y(\Gamma) + (\eta/\hat{\eta}) \pi(y, \Gamma)$$

where $\delta_y(\Gamma)$ is one if y is in Γ and zero otherwise. Thus under this alternative representation jumps are more frequent, but once a jump occurs there is state dependent positive probability that the process will stay put. Elementary computations show that if we assume that

$$\int [1/\eta(y)] d\tilde{\mathcal{Q}}(y) < \infty,$$

and construct a probability measure \mathcal{Q} to be

$$(4.3) \quad d\mathcal{Q} = \frac{d\tilde{\mathcal{Q}}}{\eta \int (1/\eta) d\tilde{\mathcal{Q}}},$$

then \mathcal{Q} is a stationary distribution for the alternative Markov chain. Therefore the Markov jump process $\{x_t\}$ will be stationary so long as it is initialized at \mathcal{Q} .

Define the conditional expectation operator $\tilde{\mathcal{T}}$ associated with the underlying Markov chain:

$$\tilde{\mathcal{T}}\phi \equiv \int \phi(y') \pi(y, dy').$$

Analogous to the operator \mathcal{T} , $\tilde{\mathcal{T}}$ maps $\mathcal{L}^2(\tilde{\mathcal{Q}})$ into itself. Using the fact that

$$\mathcal{T}_t\phi - \phi = t\eta\tilde{\mathcal{T}}\phi - t\eta\phi + o(t),$$

one can show that the generator for the continuous time jump process can be represented as

$$(4.4) \quad \mathcal{A}\phi = \eta[\tilde{\mathcal{T}}\phi - \phi]$$

(e.g., see Ethier and Kurtz (1986, pp. 162–163)). It is easy to verify that the generator \mathcal{A} is a bounded operator on all of $\mathcal{L}^2(\mathcal{Q})$. Since η is bounded, any function ϕ in $\mathcal{L}^2(\mathcal{Q})$ must also be in $\mathcal{L}^2(\tilde{\mathcal{Q}})$.

It is also of interest to characterize the adjoint \mathcal{A}^* of \mathcal{A} . To do this we study the transition probabilities for the reverse-time process. Our candidate for \mathcal{A}^* uses the adjoint $\tilde{\mathcal{T}}^*$ in place of $\tilde{\mathcal{T}}$:

$$\mathcal{A}^*\phi^* = \eta[\tilde{\mathcal{T}}^*\phi^* - \phi^*].$$

To verify that \mathcal{A}^* is the adjoint of \mathcal{A} , first note that

$$\langle \mathcal{A}\phi | \phi^* \rangle = \int \eta[\tilde{\mathcal{T}}\phi - \phi] \phi^* d\mathcal{Q} = \int \tilde{\mathcal{T}}\phi \phi^* \eta d\mathcal{Q} - \int \eta \phi \phi^* d\mathcal{Q}.$$

By construction $\tilde{\mathcal{T}}^*$ is the $\mathcal{L}^2(\tilde{\mathcal{Q}})$ adjoint of $\tilde{\mathcal{T}}$. It follows from (4.3) that $\eta d\mathcal{Q}$ is proportional to $d\tilde{\mathcal{Q}}$. Consequently,

$$\begin{aligned} \int \tilde{\mathcal{T}}\phi \phi^* \eta d\mathcal{Q} - \int \eta \phi \phi^* d\mathcal{Q} &= \int \phi \tilde{\mathcal{T}}^*\phi^* \eta d\mathcal{Q} - \int \eta \phi \phi^* d\mathcal{Q} \\ &= \langle \phi | \mathcal{A}^*\phi^* \rangle. \end{aligned}$$

To use moment conditions C1 and C2 for this Markov jump process requires that we compute $\tilde{\mathcal{T}}\phi$ and $\tilde{\mathcal{T}}^*\phi$ for test functions ϕ . Suppose that the Markov chain is a discrete time Gaussian process. It is then straightforward to evaluate $\tilde{\mathcal{T}}\phi$ and $\tilde{\mathcal{T}}^*\phi$ for polynomial test functions. Nonlinearities in the continuous-time process could still be captured by nonlinear specification of the function η . On the other hand, when nonlinearities are introduced into the specification of the Markov chain, it may be difficult to compute $\tilde{\mathcal{T}}\phi$ and $\tilde{\mathcal{T}}^*\phi$. In these cases our approach may not be any more tractable than, say, the method of maximum likelihood.

Recall that the moment conditions C1 and C2 can be used in situations in which the sampling interval is fixed and hence where the econometrician does not know the number of jumps that occurred between observations. This should be contrasted with econometric methods designed to exploit the duration time in each state.

5. OBSERVABLE IMPLICATIONS

Recall that in Section 3 we derived two sets of moment conditions to be used in discriminating among a family of candidate generators. In this section we study the informational content of these two sets of moment conditions. Formally, there is a true generator \mathcal{A} underlying the discrete-time observations. We then characterize the class of observationally equivalent generators from the vantage point of each of the sets of moment conditions. In the subsequent discussion, when we refer to a *candidate generator* we will presume that it is a generator for a Markov process.

We will establish the following identification results. First we will show that if a candidate generator $\hat{\mathcal{A}}$ satisfies the moment conditions in the set C1, it has a stationary distribution in common with the true process. Second we will show that if $\hat{\mathcal{A}}$ also satisfies the moment conditions in the set C2, it must commute with the true conditional expectation operator \mathcal{T} . If in addition $\hat{\mathcal{A}}$ is self adjoint,

and the true process is reversible (i.e. \mathcal{A} is self adjoint), then $\hat{\mathcal{A}}$ and \mathcal{A} commute. One implication of this last result is that the drift and diffusion coefficients of stationary scalar diffusions can be identified up to a common scale factor using C1 and C2. A byproduct of our analysis is that using the conditional expectation operator allows one to identify fully the generator of reversible processes.

Consider first moment conditions C1. Since one of the goals of the econometric analysis is to ascertain whether a candidate generator $\hat{\mathcal{A}}$ has \mathcal{Q} as a stationary distribution, it is preferable to begin with a specification of $\hat{\mathcal{A}}$ without reference to $\mathcal{L}^2(\mathcal{Q})$. Instead let \mathcal{B} denote the space of bounded functions on \mathbf{R}^n endowed with the sup norm. Suppose that $\hat{\mathcal{A}}$ is the infinitesimal generator for a strongly continuous contraction semigroup $\{\hat{\mathcal{T}}_t : t \geq 0\}$ defined on a closed subspace \mathcal{L} of \mathcal{B} containing at least all of the continuous functions with compact support. In this setting, strong continuity is the sup-norm counterpart to Assumption A1, i.e. $\{\hat{\mathcal{T}}_t \phi : t \geq 0\}$ converges uniformly to ϕ as t declines to zero for all ϕ in \mathcal{L} . Let $\hat{\mathcal{D}}$ denote the domain of $\hat{\mathcal{A}}$. We say that a candidate $\hat{\mathcal{A}}$ has \mathcal{Q} as its stationary distribution if \mathcal{Q} is the stationary distribution for a Markov process associated with this candidate. The following result is very similar to part of Proposition 9.2 of Ethier and Kurtz (1986, p. 239).

PROPOSITION 1: *Let $\hat{\mathcal{A}}$ be a candidate generator defined on $\hat{\mathcal{D}} \subseteq \mathcal{L}$. Then $\hat{\mathcal{A}}$ satisfies C1 for all $\phi \in \hat{\mathcal{D}}$ if and only if \mathcal{Q} is a stationary distribution of $\hat{\mathcal{A}}$.*

PROOF: Since convergence in the sup norm implies $\mathcal{L}^2(\mathcal{Q})$ convergence, our original derivation of C1 still applies. Conversely, note that the analog of Property P4 implies that for any $\phi \in \mathcal{L} : \int_0^t \hat{\mathcal{T}}_s(\phi) ds \in \hat{\mathcal{D}}$ and

$$(5.1) \quad \hat{\mathcal{A}} \int_0^t \hat{\mathcal{T}}_s \phi ds = \hat{\mathcal{T}}_t(\phi) - \phi.$$

Hence integrating both sides of (5.1) with respect to \mathcal{Q} and using the fact that $\hat{\mathcal{A}}$ satisfies C1, we have that for all $\phi \in \mathcal{L}$:

$$(5.2) \quad \int_{\mathbf{R}^n} (\hat{\mathcal{T}}_t \phi - \phi) d\mathcal{Q} = 0.$$

Relation (5.2) can be shown to hold for all indicator functions of Borelians of \mathbf{R}^n because \mathcal{L} contains all continuous functions with a compact support. *Q.E.D.*

In light of Proposition 1, any infinitesimal generator $\hat{\mathcal{A}}$ satisfying C1 has \mathcal{Q} as a stationary distribution. In other words, moment conditions in the family C1 cannot be used to discriminate among models with stationary distribution \mathcal{Q} . On the other hand, if $\hat{\mathcal{A}}$ does not have \mathcal{Q} as its stationary distribution, then there exists a test function ϕ in $\hat{\mathcal{D}}$ such that $E \hat{\mathcal{A}} \phi$ is different from zero.

EXAMPLES 1 AND 2 (Continued): Suppose $\{x_t\}$ is a scalar diffusion that satisfies equation (2.3). If the stationary density q is given by the right-hand side of (2.4), then

$$(5.3) \quad \mu = (1/2)[(\sigma^2)' + \sigma^2 q'/q].$$

For a fixed q , this equation relates the diffusion coefficient σ^2 with the local drift μ . This equation gives us sets of observationally equivalent pairs (μ, σ^2) from the vantage point of C1. In fact Banon (1978) and Cobb, Koopstein, and Chen (1983) used equation (5.3) as a basis to construct flexible (nonparametric) estimators of μ for prespecified σ^2 . As is evident from (5.3) parameterizing (μ, σ^2) is equivalent, modulo some invertibility and regularity conditions, to parameterizing $(q'/q, \sigma^2)$. Sometimes the latter parameterization is simpler and more natural. If we start by describing our candidate models using this parameterization, moment conditions C1 yield no information about the diffusion coefficient. On the other hand, as we will see later in this section, moment conditions C2 provide a considerable amount of information about the diffusion coefficient.

To illustrate these points, as in Cobb, Koopstein, and Chen (1983), consider a family of diffusions defined on the nonnegative reals parameterized by a "truncated" Laurent series:

$$(q'/q)(y, \alpha) = \sum_{j=-k}^l \alpha_j y^j,$$

where $\alpha = (\alpha_{-k}, \dots, \alpha_l)$ is a vector of unknown parameters that must satisfy certain restrictions for q to be nonnegative and integrable. We also assume that $\sigma^2 = \rho y^\gamma$ where $\rho > 0$ and $\gamma > 0$. This parameterization is sufficiently rich to encompass the familiar "square-root" process used in the bond pricing literature as well as other processes that exhibit other volatility elasticities. The implicit parameterization of μ can be deduced from (5.3).¹⁰ Moment conditions C1 will suffice for the identification of α . Since for fixed α , variations in γ leave invariant the stationary distribution q , γ cannot be inferred from moment conditions C1. However, as we will see in our subsequent analysis, moment conditions C2 will allow us to identify γ , but not ρ .

To assess the incremental informational content of the set of moment conditions C2, we focus only on generators that satisfy C1. In light of Proposition 1, all of these candidates have \mathcal{Q} as a stationary distribution. Strong continuity of the semigroup $\{\hat{\mathcal{G}}_t : t \geq 0\}$ in \mathcal{L} implies Assumption A1. Thus we are now free to use $\mathcal{L}^2(\mathcal{Q})$ (instead of the more restrictive domain \mathcal{L}) as the common domain of the semigroups associated with the candidate generators. To

¹⁰ Additional restrictions must be imposed on the parameters to guarantee that there is a solution to the associated stochastic differential equation. As mentioned in footnote 6, there are a variety of alternative sufficient conditions that can be employed to ensure that a solution exists. Alternatively, we can work directly with the implied infinitesimal generators and verify that there exist associated Markov processes for the admissible parameter values.

avoid introducing new notation, for a candidate generator $\hat{\mathcal{A}}$ satisfying C1, we will still denote by $\hat{\mathcal{A}}$ the generator of the semigroup defined on $\mathcal{L}^2(\mathcal{Q})$ and by $\hat{\mathcal{D}}$ its domain.

Recall that C2 was derived using the fact that \mathcal{A} and \mathcal{T} commute. In fact, if a candidate $\hat{\mathcal{A}}$ satisfies C2, then $\hat{\mathcal{A}}$ must commute with \mathcal{T} , i.e., for any ϕ in $\hat{\mathcal{D}}$, $\mathcal{T}\phi$ is in $\hat{\mathcal{D}}$ and $\hat{\mathcal{A}}\mathcal{T}\phi = \mathcal{T}\hat{\mathcal{A}}\phi$.

PROPOSITION 2: *Suppose $\hat{\mathcal{A}}$ satisfies C1. Then $\hat{\mathcal{A}}$ satisfies C2 for $\phi \in \hat{\mathcal{D}}$ and $\phi^* \in \hat{\mathcal{D}}^*$ if, and only if, $\hat{\mathcal{A}}\mathcal{T} = \mathcal{T}\hat{\mathcal{A}}$.*

PROOF: By mimicking the reasoning in Section 3 one shows that $\hat{\mathcal{A}}\mathcal{T} = \mathcal{T}\hat{\mathcal{A}}$ is sufficient for C2. To prove necessity, note that for any ϕ in $\hat{\mathcal{D}}$ and any ϕ^* in $\hat{\mathcal{D}}^*$, it follows from C2 and the Law of Iterated Expectations that

$$(5.4) \quad \langle \mathcal{T}\hat{\mathcal{A}}\phi | \phi^* \rangle = \langle \phi | \mathcal{T}^*\hat{\mathcal{A}}^*\phi^* \rangle.$$

Since \mathcal{T} is the adjoint of \mathcal{T}^* ,

$$(5.5) \quad \langle \phi | \mathcal{T}^*\hat{\mathcal{A}}^*\phi^* \rangle = \langle \mathcal{T}\phi | \hat{\mathcal{A}}^*\phi^* \rangle.$$

It follows that $\hat{\mathcal{A}}\mathcal{T}\phi = \mathcal{T}\hat{\mathcal{A}}\phi$ for all ϕ in $\hat{\mathcal{D}}$, because the adjoint of $\hat{\mathcal{A}}^*$ is $\hat{\mathcal{A}}$. Q.E.D.

Since it is typically hard to compute $\mathcal{T}\phi$ for an arbitrary function ϕ in $\mathcal{L}^2(\mathcal{Q})$, it may be difficult to establish directly that $\hat{\mathcal{A}}$ commutes with \mathcal{T} . As an alternative, it is often informative to check whether $\hat{\mathcal{A}}$ commutes with \mathcal{A} . To motivate this exercise, we investigate moment conditions C2 for arbitrarily small sampling intervals. By Proposition 2, this is equivalent to studying whether \mathcal{T}_t and $\hat{\mathcal{A}}$ commute for arbitrarily small t .

PROPOSITION 3: *Suppose $\hat{\mathcal{A}}$ commutes with \mathcal{T}_t for all sufficiently small t . Then $\hat{\mathcal{A}}\mathcal{A}\phi = \mathcal{A}\hat{\mathcal{A}}\phi$ for all ϕ in $\hat{\mathcal{D}}$ with $\mathcal{A}\phi$ in \mathcal{D} .*

PROOF: Note that

$$(5.6) \quad \hat{\mathcal{A}}\mathcal{A}\phi = \lim_{t \downarrow 0} [(\mathcal{T}_t - I)/t] \hat{\mathcal{A}}\phi = \lim_{t \downarrow 0} \hat{\mathcal{A}} [(\mathcal{T}_t - I)/t] \phi.$$

Since $\hat{\mathcal{A}}$ has a closed graph, the right side of (5.6) must converge to $\hat{\mathcal{A}}\mathcal{A}\phi$. Q.E.D.

In light of this result, we know that any $\hat{\mathcal{A}}$ satisfying the *small interval* counterpart to C2 must commute with \mathcal{A} . Thus from Propositions 2 and 3, if there exists an admissible test function ϕ such that

$$\hat{\mathcal{A}}\mathcal{A}\phi \neq \mathcal{A}\hat{\mathcal{A}}\phi,$$

then there is a sampling interval for which $\hat{\mathcal{A}}$ fails to satisfy some of the moment conditions in the collection C2. While we know there exists such a sampling

interval, the conclusion is not necessarily applicable to all sampling intervals and hence may not be applicable to one corresponding to the observed data. As we will now see, this limitation can be overcome when additional restrictions are placed on the candidate and true generators. Conversely, suppose that $\hat{\mathcal{A}}$ commutes with \mathcal{A} in a subspace of $\mathcal{L}^2(\mathcal{E})$. Bequillard (1989) gave sufficient conditions on that subspace to ensure that $\hat{\mathcal{A}}$ commutes with $\{\mathcal{T}_t : t \geq 0\}$. Hence such a candidate generator can never be distinguished from \mathcal{A} using moment conditions C2.

When both the candidate and true processes are reversible, requiring that $\hat{\mathcal{A}}$ commute with \mathcal{T} is equivalent to requiring that $\hat{\mathcal{A}}$ commute with \mathcal{A} . To verify this equivalence, recall that reversible Markov processes have infinitesimal generators that are self adjoint. Such operators have unique spectral representations of the following form:

$$(5.7) \quad \mathcal{A}\phi = \int_{(-\infty, 0]} \lambda d\mathcal{E}(\lambda)\phi$$

where \mathcal{E} is a “resolution of the identity”:

DEFINITION: \mathcal{E} is a *resolution of the identity* if

- (i) $\mathcal{E}(\Lambda)$ is a self-adjoint projection operator on $\mathcal{L}^2(\mathcal{E})$ for any Borelian $\Lambda \subset \mathbf{R}$;
- (ii) $\mathcal{E}(\Phi) = 0, \mathcal{E}(\mathbf{R}) = I$;
- (iii) for any two Borelians Λ_1 and $\Lambda_2, \mathcal{E}(\Lambda_1 \cap \Lambda_2) = \mathcal{E}(\Lambda_1)\mathcal{E}(\Lambda_2)$;
- (iv) for any two disjoint Borelians Λ_1 and $\Lambda_2, \mathcal{E}(\Lambda_1 \cup \Lambda_2) = \mathcal{E}(\Lambda_1) + \mathcal{E}(\Lambda_2)$;
- (v) for $\phi \in \mathcal{D}$ and $\psi \in \mathcal{L}^2(\mathcal{E}), \langle \mathcal{E}\phi | \psi \rangle$ defines a measure on the Borelians.

Spectral representation (5.7) is the operator counterpart to the spectral representation of symmetric matrices. It gives an orthogonal decomposition of the operator \mathcal{A} in the sense that if Λ_1 and Λ_2 are disjoint, $\mathcal{E}(\Lambda_1)$ and $\mathcal{E}(\Lambda_2)$ project onto orthogonal subspaces (see condition (iv) characterizing the resolution of the identity). When the spectral measure of \mathcal{E} has a mass point at a particular point λ , then λ is an eigenvalue of the operator and $\mathcal{E}(\lambda)$ projects onto the linear space of eigenfunctions associated with that eigenvalue. In light of condition (v), the integration in (5.7) can be defined formally in terms of inner products. The integration can be confined to the interval $(-\infty, 0]$ instead of all of \mathbf{R} because \mathcal{A} is negative semidefinite (e.g., see Theorems 13.30 and 13.31 on p. 349 of Rudin (1973)). Finally, spectral decomposition (5.7) of \mathcal{A} permits us to represent the semigroup $\{\mathcal{T}_t : t \geq 0\}$ via the exponential formula

$$(5.8) \quad \mathcal{T}_t\phi = \int_{(-\infty, 0]} \exp(\lambda t) d\mathcal{E}(\lambda)\phi$$

(e.g., see Theorem 13.37 of Rudin (1973, p. 360)).

Consider now a candidate generator $\hat{\mathcal{A}}$ that is reversible and satisfies C1. Then

$$(5.9) \quad \hat{\mathcal{A}}\phi = \int_{(-\infty, 0]} \lambda d\hat{\mathcal{E}}(\lambda)\phi,$$

where $\hat{\mathcal{E}}$ is also resolution of the identity. Suppose that $\hat{\mathcal{A}}$ commutes with \mathcal{T} . Then \mathcal{T} must commute with $\hat{\mathcal{E}}(\Lambda)$ for every Borelian Λ (see Theorem 13.33 of Rudin (1973, p. 351)). Let $\{\hat{\mathcal{T}}_t\}$ be the semigroup generated by $\hat{\mathcal{A}}$. Since $\hat{\mathcal{T}}_\tau$ is a bounded operator and can be constructed from $\hat{\mathcal{A}}$ via the exponential formula analogous to (5.8), $\hat{\mathcal{T}}_\tau$ must commute with \mathcal{T} for every nonnegative τ . Moreover, $\hat{\mathcal{T}}_\tau$ must commute with $\mathcal{E}(\Lambda)$ for any Borelian Λ and hence with \mathcal{T}_t for any nonnegative t . We have thus established the following proposition.

PROPOSITION 4: *Suppose $\hat{\mathcal{A}}$ satisfies C1 and \mathcal{A} and $\hat{\mathcal{A}}$ are self-adjoint. If $\hat{\mathcal{A}}\mathcal{T} = \mathcal{T}\hat{\mathcal{A}}$, then $\hat{\mathcal{A}}\mathcal{T}_t = \mathcal{T}_t\hat{\mathcal{A}}$ for all $t \geq 0$.*

Propositions 1–4 support the following approach to identification. Suppose one begins with a parameterization of a family of candidate generators. First partition this family into collections of generators with the same stationary distributions. Then partition further the family into groups of generators that commute (on a sufficiently rich collection of test functions). Two elements in the same subset of the original partition cannot be distinguished on the basis of moment conditions in the set C1, and two elements in the same subset of the finest partition cannot be distinguished on the basis of C2.

Moment conditions C1 and C2 do not capture all of the information from discrete-time data pertinent to discriminating among generators. To see this, we consider identification results based instead on knowledge of the discrete-time conditional expectation operator \mathcal{T} . In the case of reversible Markov processes, any candidate $\hat{\mathcal{A}}$ that implies \mathcal{T} as a conditional expectation operator, must satisfy the exponential formula:

$$(5.10) \quad \mathcal{T}\phi = \int_{-\infty}^0 \exp(\lambda t) d\hat{\mathcal{E}}(\lambda)\phi.$$

Moreover, the spectral decomposition for self-adjoint operators is unique (see Theorem 13.30, Rudin (1973, p. 348)). Since the exponential function is one-to-one, it follows that $\hat{\mathcal{E}}$ and \mathcal{E} and hence $\hat{\mathcal{A}}$ and \mathcal{A} must coincide. Thus we have shown the following proposition.

PROPOSITION 5: *Suppose that the generators \mathcal{A} and $\hat{\mathcal{A}}$ are self-adjoint and imply the same conditional expectation operator \mathcal{T} . Then $\mathcal{A} = \hat{\mathcal{A}}$.*

Proposition 5 implies that there is no *aliasing* problem when the Markov processes is known to be reversible. Aliasing problems in Markov processes arise because of the presence of complex eigenvalues of the generators. For reversible processes all of the eigenvalues are real and negative (the correspond-

ing resolutions of the identity are concentrated on the nonpositive real numbers). Similarly, the eigenvalues of \mathcal{F} must be in the interval $(0, 1]$. As we will see in our examples, moment conditions C1 and C2 fail to achieve complete identification of \mathcal{A} for reversible processes.

More generally, even if we do not impose reversibility, $\hat{\mathcal{A}}$ and \mathcal{F} are connected through the following alternative exponential formula:

$$(5.11) \quad \lim_{n \rightarrow \infty} (I - \hat{\mathcal{A}}/n)^{-n} \phi = \mathcal{F}\phi \quad \text{for all } \phi \in \mathcal{L}^2(\mathcal{Q})$$

(e.g., see Pazy (1983, p. 33)). This exponential formula is the generalization of a formula used to study *aliasing* and *embeddability* for finite state Markov chains (e.g., see Johansen (1973); Singer and Spillerman (1976)). Since $\hat{\mathcal{A}}$ is a continuous operator in this case, the exponential formulas simplify to $\mathcal{F} = \exp(\hat{\mathcal{A}})$. Relation (5.11) also encompasses the exponential formulas derived by Phillips (1973) and Hansen and Sargent (1983) in their analysis of aliasing in the class of multivariate Gaussian diffusion models.

We now apply and illustrate our results in the context of three examples.

EXAMPLES 1 AND 2 (Continued): As we saw earlier, using moment conditions in the set C1 may still leave a large class of observationally equivalent Markov processes. It is also clear that the two families of moment conditions cannot distinguish between two diffusions with generators that are equal up to a scale factor. We now show that using moment conditions C2 we can in fact identify the diffusion up to scale.

Suppose that both the true and the candidate processes are reversible scalar diffusions that share a stationary distribution \mathcal{Q} . Let μ and σ^2 denote the local mean and diffusion coefficients associated with the true process, and let $\hat{\mu}$ and $\hat{\sigma}^2$ denote their counterparts for the candidate process. We maintain the assumptions made in Example 1 of Section 2. Write L for the second-order differential operator associated with the pair (μ, σ^2) and \hat{L} for the corresponding operator associated with $(\hat{\mu}, \hat{\sigma}^2)$.

We start by examining the case of diffusions on a compact interval $[\ell, u]$ with two reflective barriers and a strictly positive diffusion coefficient, to which the standard Sturm-Liouville theory of second-order differential equations applies. Consider the following eigenvalue problem associated with \hat{L} :

$$(5.12) \quad \hat{L}\phi = \hat{\lambda}\phi, \quad \phi'(\ell) = \phi'(u) = 0.$$

In light of the boundary conditions imposed in (5.12), we know that if ϕ is a twice continuously differentiable (\mathcal{C}^2) solution to this eigenvalue problem, then ϕ is an eigenvector for $\hat{\mathcal{A}}$, i.e., $\hat{\mathcal{A}}\phi = \hat{\lambda}\phi$. From Sturm-Liouville theory there exists an infinite sequence of negative numbers $\hat{\lambda}_0 > \hat{\lambda}_1 > \hat{\lambda}_2 > \dots$ with $\lim_{n \rightarrow \infty} \hat{\lambda}_n = -\infty$ and corresponding unique, up to constant factors, \mathcal{C}^2 functions ϕ_n such that the pair $(\hat{\lambda}_n, \phi_n)$ solves the eigenvalue problem (5.12). Furthermore, the sequence of eigenfunctions $\{\phi_n\}$ are mutually orthogonal and form a basis for $\mathcal{L}^2(\mathcal{Q})$. Choose a negative $\hat{\lambda}_n$. Suppose that all of the moment

conditions in the class C2 are satisfied by $\hat{\mathcal{A}}$. By Proposition 4, \mathcal{T}_i and $\hat{\mathcal{A}}$ commute, and hence $\mathcal{T}\phi_n$ is also an eigenvector of $\hat{\mathcal{A}}$ associated with $\hat{\lambda}_n$. Consequently, $\mathcal{T}\phi_n$ is proportional to ϕ_n , implying that \mathcal{T} and hence \mathcal{A} must have ϕ_n as an eigenvector. Let λ_n denote the corresponding eigenvalue of \mathcal{A} . Since ϕ_n is \mathcal{E}^2 and satisfies the appropriate boundary conditions, $\mathcal{A}\phi_n$ coincides with $L\phi_n$ and hence ϕ_n satisfies the counterpart to (5.12):

$$(5.13) \quad L\phi_n = \lambda_n \phi_n.$$

Multiply this equation by q and substitute $(1/2)(\sigma^2 q)'$ for μq , to obtain

$$(1/2)(\sigma^2 q)'\phi_n' + (1/2)(\sigma^2 q)\phi_n'' = \lambda_n \phi_n q,$$

or

$$(1/2)(\sigma^2 q\phi_n')' = \lambda \phi_n q.$$

Therefore the eigenvector ϕ_n satisfies

$$(5.14) \quad \sigma^2(y)\phi_n'(y)q(y) = 2\lambda_n \int_{\rho}^y \phi_n(x)q(x) dx + C.$$

The boundary conditions on ϕ_n and condition C1 assure us that the constant C in (5.14) is in fact zero. Similarly, we conclude that

$$(5.15) \quad \hat{\sigma}^2(y)\phi_n'(y)q(y) = 2\hat{\lambda} \int_{\rho}^y \phi_n(x)q(x) dx.$$

It follows from (5.14) and (5.15) that σ^2 and $\hat{\sigma}^2$ are proportional and hence from formula (5.3) the μ and $\hat{\mu}$ are also proportional with the same proportionality factor given by the ratio of the eigenvalues. In other words, moment condition C2 permits us to identify the infinitesimal generator \mathcal{A} up to scale.

We now show how this identification result can be extended to processes defined on the whole real line, even though in this case the generator may fail to have a nonzero eigenvalue. We will proceed by considering reflexive barrier processes that approximate the original process. We assume that both μ and σ^2 are \mathcal{E}^2 functions and with $\sigma^2 > 0$. When $\hat{\mathcal{A}}$ satisfies moment conditions C1, by Proposition 1 the candidate Markov process shares a stationary density q with the true process. Suppose $\hat{\mathcal{A}}$ also satisfies moment conditions C2. For a given $k > 0$ consider the processes created by adding to the original candidate and actual processes reflexive barriers at $-k$ and k . The reflexive barrier processes will share a common stationary distribution \mathcal{Q}_k with a density q_k that, in the interval $[-k, k]$, is proportional to q . We write \mathcal{A}_k and $\hat{\mathcal{A}}_k$ for the infinitesimal generators associated with the reflexive barrier processes in $\mathcal{L}^2(\mathcal{Q}_k)$. As before we can use Sturm-Liouville theory to establish the existence of a negative $\hat{\lambda}$ and a \mathcal{E}^2 function ϕ such that

$$(5.16) \quad \hat{L}\phi = \hat{\lambda}\phi, \quad \phi'(-k) = \phi'(k) = 0.$$

Since μ and σ^2 are \mathcal{E}^2 functions, and $\sigma^2 > 0$, the eigenvector $\phi \in \mathcal{E}^4$ on $(-k, k)$ and all derivatives up to fourth order have well defined limits as

$y \rightarrow \pm k$. We write $f^{(n)}$ for the n th derivative of f . Construct ψ_+ to be a \mathcal{E}^4 function defined on (k, ∞) with $\psi_+(y) = 0$ when $y \geq k + 1$ and $\lim_{y \rightarrow k} \psi_+^{(n)}(y) = \lim_{y \rightarrow k} \phi^{(n)}(y)$ for $0 \leq n \leq 4$. Similarly construct ψ_- to be a \mathcal{E}^4 function defined on $(-\infty, -k)$ with $\psi_-(y) = 0$ when $y \leq -k - 1$ and $\lim_{y \rightarrow -k} \psi_-^{(n)}(y) = \lim_{y \rightarrow -k} \phi^{(n)}(y)$ for $0 \leq n \leq 4$. Finally let $\psi(y) \equiv \phi(y)$ if $-k \leq y \leq k$, and $\psi(y) \equiv \psi_+(\psi_-)$ if $y > k$ (resp. $y < -k$).

Notice ψ is a \mathcal{E}^4 function with support in $[-k - 1, k + 1]$ and that $\hat{L}\psi$ is \mathcal{E}^2 and has a compact support. Hence $\mathcal{A}\psi = \hat{L}\psi$ and $\mathcal{A}\hat{\mathcal{A}}\psi = L\hat{L}\psi$. Also, $L\psi$ is a \mathcal{E}^2 function with compact support and hence $\hat{\mathcal{A}}L\psi = \hat{L}L\psi$. Since by Propositions 3 and 4, \mathcal{A} and $\hat{\mathcal{A}}$ must commute, $\hat{L}L\psi = L\hat{L}\psi$, and hence $\hat{\mathcal{A}}_k\mathcal{A}_k\phi = \mathcal{A}_k\hat{\mathcal{A}}_k\phi$. The result for the compact support case with reflexive barriers implies that the drift and diffusion coefficient are identified up to scale in an interval $(-k, k)$ for arbitrary k .

Finally, we observe that if an eigenvector ϕ of \mathcal{A} can be explicitly calculated, then the corresponding (real) eigenvalue and hence scale constant can also be identified by using the fact that $\mathcal{T}\phi = e^\lambda\phi$. Alternatively, suppose it is known that the spectral measure of \mathcal{A} is concentrated at mass points with zero as an isolated mass point with multiplicity one.¹¹ Further suppose that $\hat{\mathcal{A}}$ is a candidate generator that is known to be proportional to \mathcal{A} . The smallest (in absolute value) nonzero eigenvalue $\hat{\lambda}$ of $\hat{\mathcal{A}}$ is given by

$$\hat{\lambda} \equiv \max_{\int \phi d\mathcal{E} = 0, \|\phi\| = 1} \langle \phi | \hat{\mathcal{A}}\phi \rangle.$$

Under ergodicity restrictions (see Section 7), the criterion for the maximization problem and constraints can be approximated by forming time series averages of the appropriate functions of the data. Similarly, the smallest (in absolute value) nonzero of \mathcal{A} is the logarithm of the largest eigenvalue different from one of \mathcal{T} and hence can be represented as

$$\lambda \equiv \log \max_{\int \phi d\mathcal{E} = 0, \|\phi\| = 1} \langle \phi | \mathcal{T}\phi \rangle.$$

Using the Law of Iterated Expectations, the criterion for this maximization problem can be expressed as $E\phi(x_t)\phi(x_{t+1})$. Again the criterion and the constraints can be approximated by time series averages. The true generator can be identified via

$$\mathcal{A} = \frac{\lambda}{\hat{\lambda}} \hat{\mathcal{A}}.$$

Since the constraint set for the two maximization problems of interest is infinite-dimensional, it is not trivial to go from this identification scheme to a formal justification of an estimation method. While such an exercise is beyond the scope of this paper, some of the results in Section 7 should be helpful.

¹¹ In Section 7 we will present sufficient conditions for zero to be an isolated point with multiplicity one.

EXAMPLE 3 (Continued): In Section 4 we introduced a class of factor models in which the process $\{x_t\}$ is a time invariant function of a vector of independent scalar diffusions. Suppose that both the candidate process and the true process satisfy these factor restrictions. Then it follows from Propositions 1–4 that \mathcal{A} and $\hat{\mathcal{A}}$ can be distinguished on the basis of our moment conditions if they imply different stationary distributions or if they fail to commute. By extending the identification discussion of Examples 1 and 2, we can construct $\hat{\mathcal{A}}$'s that are not distinguishable from \mathcal{A} as follows. Use the same function F mapping the factors into the observable processes as is used for \mathcal{A} . Then form an $\hat{\mathcal{A}}$ by multiplying the coefficients of each scalar factor diffusion by a possibly distinct scale factor. While such an $\hat{\mathcal{A}}$ cannot be distinguished using moment conditions C1 and C2, they can be distinguished based on knowledge of the conditional expectation operator \mathcal{T} (see Proposition 5).

In concluding this section, we illustrate an additional identification problem for our moment conditions. Starting from an arbitrary infinitesimal generator \mathcal{A} , we can always construct a two-parameter family of candidate generators that always satisfy C1 and C2. Let $\{x_t\}$ be a continuous time stationary Markov process with infinitesimal generator \mathcal{A} on a domain \mathcal{D} . For any $\phi \in \mathcal{D}$, construct $\hat{\mathcal{A}}\phi = \eta_1\mathcal{A}\phi + \eta_2[\int \phi d\mathcal{Q} - \phi]$ where η_1 and η_2 are two positive real numbers. Notice that we formed our candidate $\hat{\mathcal{A}}$ by changing the *speed* of the original process by multiplying \mathcal{A} by η_1 and adding to that the generator for a particular Markov jump process (see (4.4)). It is easy to check that $\hat{\mathcal{A}}$ has \mathcal{Q} as its stationary distribution and commutes with \mathcal{A} . Therefore $\hat{\mathcal{A}}$ cannot be distinguished from \mathcal{A} using C1 and C2. Moreover, when \mathcal{A} is self adjoint so is $\hat{\mathcal{A}}$. Consequently it follows from Proposition 5 that in this case the generators can be distinguished based on knowledge of the conditional expectations operator.

6. CORES

So far, we have analyzed observable implications by assuming all of the moment conditions in C1 or C2 could be checked. To perform such a check would require both knowledge of the domain $\hat{\mathcal{S}}$ and the ability to compute $\hat{\mathcal{A}}$. It is often difficult to characterize the domain $\hat{\mathcal{S}}$ and to evaluate a candidate generator applied to an arbitrary element of that domain. For instance, in the scalar diffusion example (Example 1) we only characterized the infinitesimal generator on a subset of the domain. Furthermore, it is desirable to have a *common* set of test functions to use for a parameterized family of generators.

Since the operator $\hat{\mathcal{A}}$ is not necessarily continuous, a dense subset of $\hat{\mathcal{S}}$ does not need to have a dense image under $\hat{\mathcal{A}}$. Consequently, in examining moment conditions C1 and C2, if we replace $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^*$ by arbitrary dense subsets, we may weaken their implications. In addition, recall that the moment conditions in C2 require looking at means of random variables of the form $\hat{\mathcal{A}}\phi(x_{t+1})\phi^*(x_t)$

– $\phi(x_{t+1})\hat{\mathcal{A}}^*\phi^*(x_t)$, which are differences of products of random variables with finite second moments. To ensure that standard central limit approximations work, it is convenient to restrict ϕ and ϕ^* so that $\hat{\mathcal{A}}\phi(x_{t+1})\phi^*(x_t) - \phi(x_{t+1})\hat{\mathcal{A}}^*\phi^*(x_t)$ has a finite second moment. For instance, this will be true when, in addition, both ϕ and ϕ^* are bounded.

To deal with these matters, we now describe a strategy for reducing the sets $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}^*$ that avoids losing information and results in random variables with finite second moments. The approach is based on the concept of a *core* for a generator \mathcal{A} . Recall that the graph of \mathcal{A} restricted to a set \mathcal{N} is $\{(\phi, \mathcal{A}\phi) : \phi \in \mathcal{N}\}$.

DEFINITION: A subspace \mathcal{N} of \mathcal{D} is a *core* for \mathcal{A} if the graph of \mathcal{A} restricted to \mathcal{N} is dense in the graph of \mathcal{A} .

Clearly, if \mathcal{N} is a core for \mathcal{A} , $\mathcal{A}(\mathcal{N})$ is dense in $\mathcal{A}(\mathcal{D})$. As we will argue below, in checking moment conditions C1 and C2 it suffices to look at sets whose linear spans are cores for $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^*$.

For a Markov jump model (Example 5), a candidate generator $\hat{\mathcal{A}}$ is a bounded operator. In this case, it suffices to look at a countable collection of bounded functions whose linear span is dense in $\mathcal{L}^2(\mathcal{Q})$ for all probability measures \mathcal{Q} .

For a Markov diffusion model (Examples 1 to 4), $\hat{\mathcal{A}}$ is no longer a bounded operator. In order to apply Proposition 1, it is important to characterize a core for the candidate generator defined on \mathcal{L} . Recall that we assumed that \mathcal{L} contained all the continuous functions with a compact support. For concreteness we now assume that $\mathcal{L} = \{\phi : \mathbf{R}^n \rightarrow \mathbf{R} \text{ s.t. } \phi \text{ is continuous, and } \lim_{|x| \rightarrow \infty} \phi(x) = 0\}$ with the sup norm. Notice that since convergence in the sup norm implies convergence of the mean, it suffices to verify moment conditions C1 on a core. Ethier and Kurtz (1986, Theorem 2.1, p. 371) showed that under the conditions stated in Example 1, the space of all infinitely differentiable functions with a compact support (\mathcal{E}_K^∞) forms a core for the infinitesimal generator associated with the scalar stochastic differential equation. In this case since $\mathcal{E}_K^2 \subset \mathcal{D}$, it is also a core for \mathcal{A} . Since, in analogy to the result we presented in Example 1, for these functions the infinitesimal generator is given by the second-order differential operator L , we can easily perform the calculations needed to apply moment condition C1. The Ethier and Kurtz result also covers certain cases with finite support and inaccessible or reflexive boundary conditions. Extensions to the multi-dimensional case, that typically require stronger smoothness conditions, are given in Theorem 2.6 and Remark 2.7, p. 374 of Ethier and Kurtz (1986).

To apply Propositions 2–4 we need to consider candidate generators defined on $\mathcal{L}^2(\mathcal{Q})$. If \mathcal{N} is a core for both \mathcal{A} and \mathcal{A}^* and our second moment condition holds for any $\phi \in \mathcal{N}$ and $\phi^* \in \mathcal{N}$, then it must hold for any pair $(\phi, \phi^*) \in \mathcal{D} \times \mathcal{D}^*$. This follows since if $\phi_n \rightarrow \phi$, $\hat{\mathcal{A}}\phi_n \rightarrow \hat{\mathcal{A}}\phi$, $\phi_n^* \rightarrow \phi^*$, and $\hat{\mathcal{A}}^*\phi_n^* \rightarrow \hat{\mathcal{A}}^*\phi^*$, then

since \mathcal{F} is continuous,

$$\langle \mathcal{F}\hat{\mathcal{A}}\phi_n | \phi_n^* \rangle - \langle \hat{\mathcal{A}}^*\phi_n^* | \mathcal{F}\phi_n \rangle \rightarrow \langle \mathcal{F}\hat{\mathcal{A}}\phi | \phi^* \rangle - \langle \hat{\mathcal{A}}^*\phi^* | \mathcal{F}\phi \rangle.$$

Hence it suffices to characterize cores for \mathcal{A} and \mathcal{A}^* . We can readily extend the results available in the literature on cores of infinitesimal generators for semigroups defined on subspaces of the space of continuous bounded functions with the sup norm. For instance consider an extension of Ethier and Kurtz's Theorem 2.1 cited above. We use the fact that \mathcal{N} is a core for \mathcal{A} if and only if both \mathcal{N} and the image of \mathcal{N} under $\lambda I - \mathcal{A}$ for some $\lambda > 0$ are dense on the domain of the contraction semigroup.¹² For diffusions with continuous coefficients the infinitesimal generator coincides on \mathcal{E}_K^2 with the second-order differential operator. This is true whether the semigroup is defined on \mathcal{L} or on $\mathcal{L}^2(\mathcal{Q})$. Since \mathcal{E}_K^∞ forms a core for the candidate infinitesimal generator $\hat{\mathcal{A}}_{\mathcal{Q}}$ of the semigroup defined on \mathcal{L} , there must exist a $\lambda > 0$ such that the image of \mathcal{E}_K^∞ under $\lambda I - \hat{\mathcal{L}}$ is dense in \mathcal{L} . It follows that the image of \mathcal{E}_K^∞ under $\lambda I - \hat{\mathcal{L}}$ is dense in $\mathcal{L}^2(\mathcal{Q})$, because \mathcal{E}_K^∞ is dense in $\mathcal{L}^2(\mathcal{Q})$ and sup-norm convergence implies $\mathcal{L}^2(\mathcal{Q})$ convergence. Hence \mathcal{E}_K^∞ is a core for $\hat{\mathcal{A}}$, the infinitesimal generator for the semigroup defined in $\mathcal{L}^2(\mathcal{Q})$. Notice that, since sup-norm convergence implies $\mathcal{L}^2(\mathcal{Q})$ convergence, we may apply exactly the same reasoning whenever we have a core \mathcal{N} for $\hat{\mathcal{A}}_{\mathcal{Q}}$ and we know that $\hat{\mathcal{A}}_{\mathcal{V}} = (\hat{\mathcal{A}}_{\mathcal{Q}})_{\mathcal{V}}$.¹³

Even when reduced to a core the observable implications of conditions C1 and C2 presuppose the use of a large set of test functions. Of course, for a finite data set, only a small (relative to the sample size) number of test functions will be used. We now obtain a reduction that can be used to support theoretical investigations in which the number of test functions can increase with the sample size such as Bierens (1990), who suggested a way of testing an infinite number of moment conditions using penalty functions, and Newey (1990) who derived results for efficient estimation by expanding the number of moment conditions as an explicit function of the sample size. Their analyses could be potentially adapted to the framework of this paper once we construct a countable collection of test functions whose *span* is a core. For the Markov jump process this reduction is easy since it suffices to choose any collection of functions with a dense span in $\mathcal{L}^2(\mathcal{Q})$. For Markov diffusions that have \mathcal{E}_K^2 as a core we may proceed as follows. Fix a positive integer N and consider the subspace of \mathcal{E}_K^2 of functions with support on $\{y: |y| \leq N\}$. This subspace is separable if we use the norm given by the maximum of the sup norm of a function and of its first two derivatives. Choose a countable dense collection for each N and take the union over positive integers N . Since \mathcal{E}_K^2 is a core, it is straightforward to show that the linear span of this union is also a core for \mathcal{A} .

¹² See Proposition 3.1, p. 17 of Ethier and Kurtz (1986).

¹³ If the assumptions concerning the bounds on the derivatives of the coefficients made in Example 2.1 are replaced by weaker polynomial growth conditions, it is still possible to show directly that \mathcal{E}_K^2 is a core for \mathcal{A} if the stationary distribution possesses moments of sufficiently high order.

7. ERGODICITY AND MARTINGALE APPROXIMATION

To use the moment conditions derived in Section 2 in econometric analyses, we must have some way of approximating expectations of functions of the Markov state vector x_t in the case of C1 and of functions of both x_{t+1} and x_t in the case of C2. As usual, we approximate these expectations by calculating the corresponding time-series averages from a discrete-time sample of finite length. To justify these approximations via a Law of Large Numbers, we need some form of ergodicity of the discrete-sampled process. In the first subsection, we investigate properties of the infinitesimal generator that are sufficient for ergodicity of both the continuous time and the discrete-time processes.

It is also of interest to assess the magnitude of the sampling error induced by making such approximations. This assessment is important for determining the statistical efficiency of the resulting econometric estimators and in making statistical inferences about the plausibility of candidate infinitesimal generators. The vehicle for making this assessment is a central limit theorem. In the second subsection we derive central limit approximations via the usual martingale approach. Again we study this problem from the vantage point of both continuous and discrete records, and we derive sufficient conditions for these martingale approximations to apply that are based directly on properties of infinitesimal generators.

7.1. Law of Large Numbers

Stationary processes in either discrete time or continuous time obey a Law of Large Numbers. However, the limit points of time series averages may not equal the corresponding expectations under the measure \mathcal{Q} . Instead these limit points are expectations conditioned on an appropriately constructed set of *invariant events* for the Markov process. Furthermore, the conditioning set for the continuous-time process may be a proper subset of the conditioning set for a fixed interval discrete-sampled process. Therefore, the limit points for the discrete record Law of Large Numbers may be different than the limit points for the continuous record Law of Large Numbers.

Invariant events for Markov processes (in continuous or discrete time) turn out to have a very simple structure. They are measurable functions of the initial state vector x_0 . Hence, associated with the invariant events for the continuous-time process $\{x_t\}$, there is a sigma algebra \mathcal{S} , contained in the Borelians of \mathbf{R}^n , such that any invariant event is of the form $A = \{x_0 \in B\}$ for some $B \in \mathcal{S}$. We can construct a conditional probability distribution \mathcal{Q}_y indexed by the initial state $x_0 = y$ such that expectations conditioned on \mathcal{S} (as a function of y) can be evaluated by integrating with respect to \mathcal{Q}_y . Imagine initializing the Markov process using \mathcal{Q}_y in place of \mathcal{Q} in our analysis where y is selected to be the observed value, of x_0 . Then under this new initial distribution the process $\{x_t\}$ remains stationary, but it is now ergodic for (\mathcal{Q}) almost all y . Therefore, by an appropriate initialization we can convert any stationary process into one that is also ergodic. Alternatively, it can be verified that the moment conditions C1 and

C2 hold conditioned on the invariant events for the continuous-time process $\{x_t\}$. Consequently, imposition of ergodicity for the continuous time process is made as a matter of convenience.

Since ergodicity of $\{x_t\}$ is connected directly to the initial distribution imposed on x_0 , it is of interest to have a criterion for checking whether an appropriate initial distribution has been selected. As we will see, such a criterion can be obtained by examining the zeros of the operator \mathcal{A} . Note that constant functions are always zeros of the operator \mathcal{A} . We omit all such functions from consideration except for the zero function by focusing attention on the following closed linear subspace:

$$\mathcal{Z}(\mathcal{D}) \equiv \left\{ \phi \in \mathcal{L}^2(\mathcal{D}) : \int_{\mathbb{R}^n} \phi d\mathcal{D} = 0 \right\}.$$

For any t , \mathcal{T}_t maps $\mathcal{Z}(\mathcal{D})$ into itself. Hence we may consider $\{\mathcal{T}_t : t \geq 0\}$ as having the domain $\mathcal{Z}(\mathcal{D})$. Furthermore, since \mathcal{D} is a linear subspace of $\mathcal{L}^2(\mathcal{D})$, it follows from P6 that $\mathcal{D} \cap \mathcal{Z}(\mathcal{D})$ is a linear subspace and is dense in $\mathcal{Z}(\mathcal{D})$.

Our first result relates the ergodicity of the continuous-time process $\{x_t\}$ to the uniqueness of the zeros of \mathcal{A} on $\mathcal{Z}(\mathcal{D})$ and is due to Bhattacharya (1982, Proposition 2.2).

PROPOSITION 6: *The process $\{x_t\}$ is ergodic if, and only if, $\mathcal{A}\phi = 0$ for $\phi \in \mathcal{D} \cap \mathcal{Z}(\mathcal{D})$ implies that $\phi = 0$.*

Next we consider ergodicity of the discrete-sampled process. For notational simplicity, we set the sampling interval to one. The discrete time counterpart to Proposition 6 can be proved by modifying appropriately the argument of Bhattacharya (1982).

PROPOSITION 7: *The sampled process $\{x_t : t = 0, 1, \dots\}$ is ergodic if and only if $\mathcal{T}\phi = \phi$ and $\phi \in \mathcal{Z}(\mathcal{D})$ imply $\phi = 0$.*

Ergodicity of the continuous-time process $\{x_t\}$ does not necessarily imply ergodicity of this process sampled at integer points in time. From equation (5.11) \mathcal{T}_t can be interpreted as the operator exponential of $t\mathcal{A}$ even though, strictly speaking, the exponential of $t\mathcal{A}$ may not be well defined. The Spectral Mapping Theorem for infinitesimal generators (e.g., see Pazy (1983, Theorem 2.4, p. 46)) states that the nonzero eigenvalues of \mathcal{T}_t are exponentials of the eigenvalues of $t\mathcal{A}$. Therefore, the only way in which \mathcal{T} can have a unit eigenvalue on $\mathcal{Z}(\mathcal{D})$ is for \mathcal{A} to have $2\pi ki$ as an eigenvalue on $\mathcal{Z}(\mathcal{D})$, where k is an integer. In other words, there must exist a pair of functions ϕ_r and ϕ_i in $\mathcal{Z}(\mathcal{D})$, at least one of which is different from zero, such that

$$\mathcal{A}(\phi_r + i\phi_i) = 2\pi ki(\phi_r + i\phi_i).$$

When \mathcal{A} has a purely imaginary eigenvalue there will always be a nondegenerate function ϕ in $\mathcal{Z}(\mathcal{Q})$ such that $\phi(x_t)$ is perfectly predictable given x_0 . In other words, there exists a nondegenerate function ϕ such that $\phi(x_t) = \mathcal{T}_t \phi(x_0)$ almost surely ($\mathcal{P}r$) for all $t \geq 0$.¹⁴ To ensure ergodicity for all sampling intervals, we must rule out all purely imaginary eigenvalues.

Generators of stationary, ergodic (in continuous time) Markov jump processes (Example 5) do not have purely imaginary eigenvalues. Suppose to the contrary that the pair (ϕ_r, ϕ_i) solves the eigenvector problem for eigenvalue $i\theta$ different from zero. Then

$$\tilde{\mathcal{T}}(\phi_r + i\phi_i) = [(i\theta/\eta) + 1](\phi_r + i\phi_i).$$

This leads to a contradiction because $|(i\theta/\eta) + 1| > 1$ and $\tilde{\mathcal{T}}$ is a weak contraction.

Recall that generators of a reversible process, including those discussed in Examples 1, 2, and 3, have only real eigenvalues. Many other multivariate stationary, ergodic (in continuous time) Markov diffusion processes (Example 4) do not have purely imaginary eigenvalues. For instance, under the conditions given in Section 4 for the existence of a unique stationary distribution, for sufficiently large t , $\phi(x_t) = \mathcal{T}_t \phi(x_0)$ almost surely ($\mathcal{P}r$) implies $\phi(x_0) = 0$ almost surely ($\mathcal{P}r$) (see Has'minskii (1980, Lemma 6.5, pp. 128 and 129)). Hence \mathcal{A} cannot have a purely imaginary eigenvalue in this case.

We now consider the moment conditions derived in Section 3. Recall that moment conditions C1 imply that $\mathcal{A}(\mathcal{D}) \subseteq \mathcal{Z}(\mathcal{Q})$. Consequently, the approximation results we obtained for functions in $\mathcal{Z}(\mathcal{Q})$ can be applied directly to justify forming finite sample approximations to the moment conditions C1.

Consider next moment conditions in the set C2. Recall from Section 3 that these conditions are of the form

$$(7.1) \quad E[\nu(x_{t+1}, x_t)] = 0$$

for functions $\nu: \mathbf{R}^{2n} \rightarrow \mathbf{R}$. The function ν was not necessarily restricted so that the random variable $\nu(x_{t+1}, x_t)$ has a finite second moment, although this random variable will always have a finite first moment. Ergodicity of the sampled process is sufficient for the sample averages to converge to zero almost surely ($\mathcal{P}r$). Under the additional restriction that the random variable $\nu(x_{t+1}, x_t)$ has a finite second moment, it follows from the Mean Ergodic Theorem that the sample averages will also converge to zero in $\mathcal{L}^2(\mathcal{P}r)$, the space of all random variables with finite second moments on the original probability space.

7.2. Central Limit Theorem

To obtain central limit approximations for discrete time Markov processes, Rosenblatt (1971) suggested restricting the operator \mathcal{T} to be a strong contrac-

¹⁴ Let the pair of functions (ϕ_r, ϕ_i) in $\mathcal{D} \cap \mathcal{Z}(\mathcal{Q})$ satisfy the eigenvalue problem for purely imaginary eigenvalue $i\theta$. Thus (ϕ_r, ϕ_i) solves the analogous eigenvalue problem with eigenvalue $\exp(i\theta t)$ for \mathcal{T} , and consequently, $\|\mathcal{T}_t \phi_r\|^2 + \|\mathcal{T}_t \phi_i\|^2 = \|\phi_r\|^2 + \|\phi_i\|^2$. Since $\|\phi_r\|^2 = E[|\phi_r(x_t) - \mathcal{T}_t \phi_r(x_0)|^2] + \|\mathcal{T}_t \phi_r\|^2 = 0$, by the contraction property $\|\phi_r\|^2 = \|\mathcal{T}_t \phi_r\|^2 = 0$, and similarly for ϕ_i .

tion on $\mathcal{Z}(\mathcal{C})$. Among other things, this limits the temporal dependence sufficiently for the discrete-sampled process to be strongly mixing (see Rosenblatt (1971, Lemma 3, p. 200)). The way in which the central limit approximations are typically obtained is through martingale approximations. Explicit characterization of these martingale approximations is of independent value for investigating the statistical efficiency of classes of generalized method of moments estimators constructed from infinite-dimensional families of moment conditions (e.g., see Hansen (1985)) such as those derived in Section 3.

Since our goal in this subsection is to deduce restrictions on \mathcal{A} that are sufficient for martingale approximations to apply, we begin by investigating martingale approximations for the original continuous time process. This will help to motivate restrictions on \mathcal{A} that are sufficient for \mathcal{F} to be a strong contraction. In studying moment conditions C1 using a continuous record, we use the standard argument of approximating the integral $\int_0^T \mathcal{A}\phi(x_t) dt$ by a martingale m_T , and applying a central limit theorem for martingales.¹⁵ For each $T \geq 0$, define:

$$(7.2) \quad m_T = -\phi(x_T) + \phi(x_0) + \int_0^T \mathcal{A}\phi(x_t) dt.$$

Then $\{m_T : T \geq 0\}$ is a martingale, relative to the filtration generated by the continuous-time process $\{x_t\}$ (e.g., see Ethier and Kurtz (1986, Proposition 1.7, p. 162)). The error in approximating $\int_0^T \mathcal{A}\phi(x_t) dt$ by m_T is just $-\phi(x_T) + \phi(x_0)$ which is bounded by $2\|\phi\|$. When scaled by $(1/\sqrt{T})$, this error clearly converges in $\mathcal{L}^2(\mathcal{P}_r)$ to zero. Consequently, a central limit theorem for $\{(1/\sqrt{T})\int_0^T \mathcal{A}\phi(x_t) dt\}$ can be deduced from the central limit theorem for a scaled sequence of martingales $\{(1/\sqrt{T})m_T\}$ (see Billingsley (1961)).

The random variable m_T has mean zero and variance

$$(7.3) \quad \begin{aligned} E[(m_T)^2] &= \sum_{j=1}^J E[(m_{T(j-1)/J} - m_{Tj/J})^2] \\ &= JE[(m_{T/J})^2] \\ &= TE \left[(J/T) \left(-\phi(x_{T/J}) + \phi(x_0) + \int_0^{T/J} \mathcal{A}\phi(x_t) dt \right)^2 \right] \end{aligned}$$

for any positive integer J since the increments of the martingale are stationary and orthogonal. Thus we are led to investigate the limit

$$(7.4) \quad \lim_{\epsilon \rightarrow 0} E \left[(1/\epsilon) \left(-\phi(x_\epsilon) + \phi(x_0) + \int_0^\epsilon \mathcal{A}\phi(x_t) dt \right)^2 \right].$$

¹⁵ A central limit theorem for diffusions on the line appears in Mandl (1968). Florens-Zmirou (1984) gave an alternative proof to Mandl's and derived a counterpart for the discretized version of some diffusions on the line. Our exposition of the central limit approximation for the continuous time record follows Bhattacharya (1982).

By the Triangle Inequality

$$(7.5) \quad \left\{ E \left[(1/\epsilon)^{1/2} \int_0^\epsilon \mathcal{A}\phi(x_t) dt \right]^2 \right\}^{1/2} \leq \int_0^\epsilon E \left[(1/\epsilon) \mathcal{A}\phi(x_t)^2 \right]^{1/2} dt = \epsilon^{1/2} \|\mathcal{A}\phi\|.$$

Thus the limit in (7.4) can be written as

$$(7.6) \quad \lim_{\epsilon \rightarrow 0} (1/\epsilon) E \left\{ [\phi(x_\epsilon) - \phi(x_0)]^2 \right\} = \lim_{\epsilon \rightarrow 0} (1/\epsilon) 2[\langle \phi | \phi \rangle - \langle \mathcal{T}_\epsilon \phi | \phi \rangle] = -2\langle \phi | \mathcal{A}\phi \rangle.$$

Taking limits on the right side of (7.3) as J gets large and substituting from (7.6), we find that

$$(7.7) \quad E[(m_T)^2] = -2T\langle \phi | \mathcal{A}\phi \rangle.$$

Therefore, the asymptotic variance for the central limit approximation is given by $-2\langle \phi | \mathcal{A}\phi \rangle$ which is always nonnegative due to the fact that \mathcal{A} is quasi negative semidefinite (Property P9).

The fact that the right side of (7.2) is a martingale guarantees that the continuous-time central limit theorem can always be applied to functions $\psi = \mathcal{A}\phi$. Note, however, that the limiting distribution is nondegenerate only when $\langle \phi | \mathcal{A}\phi \rangle < 0$.

We now investigate the discrete-time counterpart to this martingale approximation under the restriction that \mathcal{T} is a *strong contraction* on $\mathcal{X}(\mathcal{Q})$, i.e. there exists a constant $C < 1$ such that $\|\mathcal{T}\phi\| \leq C\|\phi\|$ for each $\phi \in \mathcal{X}(\mathcal{Q})$. Later we will discuss conditions on the generator \mathcal{A} that are sufficient for this property to hold. Using the strong contraction property, we will show that a martingale M_N approximates $\sum_{t=1}^N \psi(x_t)$ for ψ in $\mathcal{X}(\mathcal{Q})$. Of course, the ψ 's we are interested in are the ones constructed by applying \mathcal{A} to an element of its domain \mathcal{D} .

The strong contraction property of \mathcal{T} guarantees that $(I - \mathcal{T})$ has a bounded inverse on $\mathcal{X}(\mathcal{Q})$. Note that since

$$E[\psi(x_{t+1}) - \mathcal{T}\psi(x_t) | x_t] = 0,$$

the discrete-time process $\{M_N : N = 1, 2, \dots\}$, defined by

$$(7.8) \quad M_N \equiv \sum_{t=1}^N \left[(I - \mathcal{T})^{-1} \psi(x_t) - \mathcal{T}(I - \mathcal{T})^{-1} \psi(x_{t-1}) \right],$$

is a martingale adapted to the filtration generated by the discrete-sampled Markov process. Equivalently, we may write

$$M_N = \mathcal{T}(I - \mathcal{T})^{-1} [\psi(x_N) - \psi(x_0)] + \sum_{t=1}^N \psi(x_t),$$

which is the discrete time counterpart to (7.2) and agrees with a martingale approximation suggested by Gordin (1969). Note that the $\mathcal{L}^2(\mathcal{P}_r)$ norm of the

error in approximating the partial sum $\sum_{i=1}^N \psi(x_i)$ by the martingale M_N has a bound independent of N . This bound is uniform for ψ in the unit ball of $\mathcal{Z}(\mathcal{E})$ since

$$\left(E \left\{ \left| \mathcal{T}(I - \mathcal{T})^{-1} [\psi(x_N) - \psi(x_0)] \right|^2 \right\} \right)^{1/2} \leq 2 \| \mathcal{T}(I - \mathcal{T})^{-1}(\psi) \|$$

and \mathcal{T} and $(I - \mathcal{T})^{-1}$ are bounded operators on $\mathcal{Z}(\mathcal{E})$. Scaling by $(1/\sqrt{N})$ makes the approximation error arbitrarily small as N goes to infinity implying that central limit approximations for $\{(1/\sqrt{N})\sum_{i=1}^N \psi(x_i)\}$ can be deduced from central limit approximations for a scaled sequence of martingales $\{(1/\sqrt{N})M_N\}$ (see Billingsley (1961)). Finally, it follows from (7.8) that

$$\begin{aligned} (7.9) \quad (1/N)E(M_N^2) &= \langle (I - \mathcal{T})^{-1} \psi | (I - \mathcal{T})^{-1} \psi \rangle \\ &\quad - \langle \mathcal{T}(I - \mathcal{T})^{-1} \psi | \mathcal{T}(I - \mathcal{T})^{-1} \psi \rangle \\ &= \langle \psi | \psi \rangle + 2 \langle \psi | \mathcal{T}(I - \mathcal{T})^{-1} \psi \rangle, \end{aligned}$$

which gives the asymptotic variance for the discrete-time central limit theorem. It can be shown that for $\psi = \mathcal{A}\phi$, the expression on the right side of (7.9) is greater than or equal to the corresponding expression $-2\langle \phi | \mathcal{A}\phi \rangle$ for the continuous-time martingale approximation. This reflects the loss of information due to sampling in discrete time.

The discrete-time martingale approximation given by equation (7.8) can also be applied to moment conditions in the class C2. As we argued previously, these moment conditions can be represented as in (7.1). Suppose that the random variable $\nu(x_{t+1}, x_t)$ has a finite second moment. Then there exists a ψ in $\mathcal{Z}(\mathcal{E})$ such that

$$E[\nu(x_{t+1}, x_t) | x_t] = \psi(x_t).$$

Then a martingale approximator for $\sum_{i=1}^N \nu(x_{t+1}, x_t)$ is given by the sum of the martingale $\sum_{i=1}^N [\nu(x_{t+1}, x_t) - \psi(x_t)]$ and the martingale approximator for $\sum_{i=1}^N \psi(x_t)$.

We now consider restrictions on \mathcal{A} that are sufficient for \mathcal{T} to be a strong contraction. We write $\text{var}(\phi)$ for the variance of the random variable $\phi(x_t)$.

CONDITION G: *There exists a subspace \mathcal{N} , a core for \mathcal{A} , and a $\delta > 0$ such that $-\langle \mathcal{A}\phi | \phi \rangle \geq \delta \text{var}(\phi)$ for all $\phi \in \mathcal{N}$.*

Notice that since \mathcal{N} is a core, Condition G can be extended to any $\phi \in \mathcal{D}$. Also, since $\phi - f\phi \, d\mathcal{E} \in \mathcal{Z}(\mathcal{E})$, Condition G is equivalent to requiring

$$-\langle \mathcal{A}\phi | \phi \rangle \geq \delta \langle \phi | \phi \rangle \quad \text{for all } \phi \text{ with mean zero.}$$

PROPOSITION 8: *The operator \mathcal{T} is a strong contraction if and only if the generator \mathcal{A} satisfies Condition G.*

PROOF: As shown by Banon (1977, Lemma 3.11, p. 79), if \mathcal{T}_t is a strong contraction for any fixed $t > 0$, then the semigroup $\{\mathcal{T}_t\}$ must satisfy the *exponential inequality* for some strictly positive δ :

$$\|\mathcal{T}_t(\phi)\| \leq \exp(-\delta t)\|\phi\| \quad \text{for all } \phi \in \mathcal{X}(\mathcal{E}).$$

Hence $\mathcal{S}_t \equiv e^{\delta t}\mathcal{T}_t$ defines a contraction semigroup in $\mathcal{X}(\mathcal{E})$ with a generator $\mathcal{A} + \delta I$. By P9, Condition G holds. Conversely if Condition G holds, $\mathcal{A} + \delta I$ is a quasi negative semidefinite operator with domain $\mathcal{D} \cap \mathcal{X}(\mathcal{E})$, and such that for any $\lambda > 0$, $\lambda I - (\mathcal{A} + \delta I)$ is onto. Hence, by the Lumer-Phillips Theorem, $\mathcal{A} + \delta I$ is the generator of a contraction semigroup, $\{\mathcal{S}_t\}$, in $\mathcal{X}(\mathcal{E})$ and since $\mathcal{T}_t = e^{-\delta t}\mathcal{S}_t$, it satisfies the exponential inequality. Q.E.D.

For reversible generators, Condition G is equivalent to zero being an isolated point in the support of \mathcal{E} , the resolution of the identity used in Section 5. In particular, for diffusions with reflecting boundaries on a compact interval with strictly positive diffusion coefficient, Condition G holds. Condition G requires that the variances of the continuous-time martingale approximators (and hence the discrete-time approximators) be bounded away from zero for test functions ϕ with unit variances. In particular, when Condition G is satisfied, the central limit approximation will be nondegenerate whenever ϕ in \mathcal{D} is not constant. Also, Condition G ensures that $\|\mathcal{A}(\phi)\|$ is bounded away from zero on the unit sphere and hence \mathcal{A}^{-1} is a bounded operator on $\mathcal{A}(\mathcal{D})$.

We now study restrictions on \mathcal{A} that imply Condition G for the examples. In the case of the Markov jump process (Example 5), a sufficient condition is that η be bounded away from zero and the conditional expectation operator $\tilde{\mathcal{T}}$ on the associated chain be a strong contraction on $\mathcal{X}(\tilde{\mathcal{E}})$. To see this, first note that for $K_1 \equiv \int \phi d\tilde{\mathcal{E}}$ and $K_2 \equiv \int (1/\eta) d\tilde{\mathcal{E}}$,

$$\begin{aligned} (7.10) \quad \langle \phi | \mathcal{A}\phi \rangle &= \int \phi [\tilde{\mathcal{T}}(\phi) - \phi] \eta d\tilde{\mathcal{E}} \\ &= (1/K_2) \int \phi [\tilde{\mathcal{T}}(\phi) - \phi] d\tilde{\mathcal{E}} \\ &= (1/K_2) \int (\phi - K_1) [\tilde{\mathcal{T}}(\phi - K_1) - (\phi - K_1)] d\tilde{\mathcal{E}} \\ &\leq (1/K_2)(\gamma - 1) \int (\phi - K_1)^2 d\tilde{\mathcal{E}} \end{aligned}$$

for some $0 < \gamma < 1$ since $\tilde{\mathcal{T}}$ is a strong contraction on $\mathcal{X}(\tilde{\mathcal{E}})$. Next observe that for $\eta_\nu \equiv \inf \eta > 0$:

$$\begin{aligned} (7.11) \quad (1/K_2)(\gamma - 1) \int (\phi - K_1)^2 d\tilde{\mathcal{E}} &\leq (\gamma - 1) \int (\phi - K_1)^2 \eta d\tilde{\mathcal{E}} \\ &\leq (\gamma - 1)\eta_\nu \langle \phi | \phi \rangle, \end{aligned}$$

as long as $\phi \in \mathcal{L}(\mathcal{Q})$. The operator \mathcal{A} in conjunction with any positive δ less than $(1 - \gamma)\eta_\ell$ will satisfy Condition G.

We now turn to Markov diffusion processes (Examples 1 to 4). Bouc and Pardoux (1984) provided sufficient conditions for G that include uniformly bounding above and below the diffusion matrix Σ as well as a pointing inward condition for the drift μ . To accommodate certain examples in finance where the diffusion coefficient may vanish at the boundary, it is necessary to relax the assumptions on the diffusion coefficient. For this reason we derive here an alternative set of sufficient conditions on the line.

Consider a scalar diffusion on an interval $[\ell, u]$, allowing for $\ell = -\infty$ and $u = +\infty$. We restrict σ^2 to be positive in the open interval (ℓ, u) and require the speed density to be integrable. If a finite boundary is attainable, we assume that it is reflexive. We start by presenting an alternative characterization of Condition G for diffusions. Let \mathcal{M} be the space of twice continuously differentiable functions $\phi \in \mathcal{L}^2(\mathcal{Q})$ such that $\sigma\phi'$ and $L\phi$ are also in $\mathcal{L}^2(\mathcal{Q})$. In the case of a reflexive boundary we also add the restriction that ϕ' vanishes at the boundary.

For $\phi \in \mathcal{M}$ it follows from the characterization of the infinitesimal generator established in Example 1 and equation (2.4), that

$$-2 \left\langle \phi - \int \phi d\mathcal{Q} \middle| \mathcal{A}(\phi) \right\rangle = \int (\phi')^2 \sigma^2 d\mathcal{Q}.$$

Therefore, Condition G requires for some $\varepsilon > 0$:

$$(7.12) \quad \int (\phi')^2 \sigma^2 d\mathcal{Q} \geq \varepsilon \text{ var}(\phi) \quad \text{for all } \phi \in \mathcal{M}.$$

When \mathcal{M} is a core condition (7.12) is equivalent to Condition G. Recall that in Section 6 we discussed sufficient conditions for \mathcal{E}_K^2 and hence \mathcal{M} to be a core.¹⁶

We now derive sufficient conditions for inequality (7.12). Let $z \in (\ell, u)$ and notice that, for any ϕ in \mathcal{M} ,

$$\int [\phi - \phi(z)]^2 d\mathcal{Q} \geq \text{var}(\phi).$$

It follows from an inequality in Muckenhoupt (1972, Theorem 2) or Talenti (1969, p. 174) that there is a finite K_1 such that

$$\int_z^u [\phi - \phi(z)]^2 d\mathcal{Q} \leq K_1 \int_z^u |\phi'|^2 \sigma^2 d\mathcal{Q}$$

if, and only if,

$$(7.13) \quad \sup_{z < r < u} \mathcal{Q}(\{[r, u]\}) \int_z^r (\sigma^2 q)^{-1} dy < \infty.$$

¹⁶ Inequality (7.12) has the obvious multivariate extension.

Hence (7.12) will hold provided (7.13) and the analogous condition

$$(7.14) \quad \sup_{\ell < r < z} \mathcal{Q}\{(\ell, r]\} \int_r^z (\sigma^2 q)^{-1} dy < \infty$$

are satisfied.

Applying L'Hospital's Rule, if

$$(7.15) \quad \lim_{r \rightarrow u} \mathcal{Q}\{[r, u)\} / \sigma(r)q(r) < \infty,$$

then (7.13) holds. In particular inequality (7.15) holds if $\liminf_{r \rightarrow u} \sigma(r)q(r) > 0$. If $\lim_{r \rightarrow u} \sigma(r)q(r) = 0$, we may apply L'Hospital's Rule again to obtain as a sufficient condition

$$(7.16) \quad \lim_{r \rightarrow u} \frac{1}{\sigma'(r) + \sigma(r)q'(r)/q(r)} \text{ exists and is finite.}$$

Using equation (5.3) that relates the logarithmic derivative of the stationary density to the drift and diffusion coefficients, we may rewrite (7.16) as

$$(7.17) \quad \lim_{r \rightarrow u} \frac{\sigma(r)}{2\mu(r) - \sigma(r)\sigma'(r)} \text{ exists and is finite.}$$

The conditions at the lower boundary are exactly analogous. We summarize this discussion in a proposition:

PROPOSITION 9: *Suppose $\{x_t\}$ solves $dx_t = \mu(x_t)dt + \sigma(x_t)dW_t$ in an interval (ℓ, u) with possibly $\ell = -\infty$ and/or $u = +\infty$; σ is \mathcal{C}^1 and positive on (ℓ, u) ; and the speed density $1/\sigma^2$ is integrable. Then the following conditions are sufficient for (7.12) to hold.*

(a) *The right boundary satisfies either $\liminf_{r \rightarrow u} \sigma(r)q(r) > 0$; or $\liminf_{r \rightarrow u} \sigma(r)q(r) = 0$ and $\lim_{r \rightarrow u} \sigma(r)/[2\mu(r) - \sigma(r)\sigma'(r)]$ exists and is finite.*

(b) *The left boundary satisfies either $\liminf_{r \rightarrow \ell} \sigma(r)q(r) > 0$; or $\liminf_{r \rightarrow \ell} \sigma(r)q(r) = 0$ and $\lim_{r \rightarrow \ell} \sigma(r)/[2\mu(r) - \sigma(r)\sigma'(r)]$ exists and is finite.*

Notice that for models that are parameterized in terms q and σ^2 , condition (7.16) is easier to verify than the equivalent condition (7.17) mentioned in the proposition. Also one may, in some cases, verify directly (7.15).

The square root process $dx_t = \kappa(x_t - \bar{x})dt + \sqrt{x_t}dW_t$, with $\kappa < 0$ and $\bar{x} > 0$, is an example of a process that satisfies the sufficient conditions of Proposition 9 even though σ^2 is not bounded away from zero. A process $\{x_t\}$ that solves a stochastic differential equation in \mathbf{R}_+ , with $\sigma(y) \equiv 1$ and $\mu(y) = -(\sqrt{y})^{-1}$ for large y , is an example where the sufficient conditions of Proposition 9 do not hold. However in this case every nonnegative real number is in the spectrum of \mathcal{A} and hence Condition G fails, even though $\{x_t\}$ is mean recurrent (Bouc and Pardoux (1984, p. 378)).

The conclusion of Proposition 9 also holds for the multivariate factor models described in Example 3 when the individual factor processes satisfy the specified conditions.

8. CONCLUSION

The analysis in this paper is intended as support of empirical work aimed at assessing the empirical plausibility of particular continuous-time Markov models that arise in a variety of areas of economics. In many instances, such models are attractive because of conceptual and computational simplifications obtained by taking continuous time limits. The approach advanced in this paper can, by design, be used to study these models empirically even when it is not possible for an econometrician to approximate a continuous data record. For this reason, we focused our analysis on fixed interval sampling although our moment conditions are also applicable more generally. For instance, we could accommodate systematic patterns to the sampling, or the sampling procedure itself could be modeled as an exogenous stationary process. Both moment conditions are still satisfied, with moment condition C2 applied to adjacent observations. The observable implications we obtained extend in the obvious way. This means that our moment conditions can easily handle missing observations that occur in financial data sets due to weekends and holidays. On the other hand, such sampling schemes may alter the central limit approximations reported in Section 7.

One of the many questions left unanswered here is that of the selection of test functions in practice. For finite-dimensional parameter models one could compare the asymptotic efficiency of estimators constructed with alternative configurations of test functions along the lines of Hansen (1985). Finite sample comparisons are likely to require Monte Carlo investigations.

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Manuscript received January, 1992; final revision received July, 1994.

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